Mathematics 132 – Calculus for Physical and Life Sciences 2
Exam 3 – Review Sheet
April 15, 2008

Sample Exam Questions - Solutions

This list is much longer than the actual exam will be (to give you some idea of the range of different questions that might be asked).

I. (A) Show that for any constant c, $y = x^2 + \frac{c}{x^2}$ is a solution of the differential equation

$$y' = 4x - \frac{2}{x}y.$$

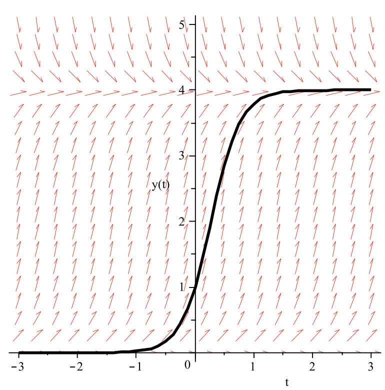
Solution: For $y = x^2 + \dfrac{c}{x^2}$ we have $y' = 2x - \dfrac{2c}{x^3}$ and $4x - \dfrac{2}{x}y = 4x - \dfrac{2}{x}(x^2 + \dfrac{c}{x^2}) = 2x - \dfrac{2c}{x^3}$. Thus $y = x^2 + \dfrac{c}{x^2}$ is a solution to the differential equation $y' = 4x - \dfrac{2}{x}y$.

(B) All parts of this question refer to the differential equation

$$y' = y(4 - y)$$

(1) Sketch the slope field of this equation, showing the slopes at points on the lines $y = 0, 1, 2, 3, 4, 5$

Solution:



(2) On your slope field, sketch the graph of the solution of the equation with $y(0) = 1$.

Solution: See figure above.

(3) Use Euler's method to approximate the solution of this equation with $y(0) = 1$ for $0 \le x \le 1$ using $n = 4$.

Solution: We have $\Delta x = 0.25$.

$$\begin{array}{ll} x_0 = 0 & y_0 = 1 \\ x_1 = .25 & y_1 = y_0 + (y_0(4-y_0))\Delta x = 1 + 3(.25) = 1.75 \\ x_2 = .5 & y_2 = y_1 + (y_1(4-y_1))\Delta x = 2.734375 \\ x_3 = .75 & y_3 = y_2 + (y_2(4-y_2))\Delta x = 3.599548340 \\ x_4 = 1 & y_4 = y_3 + (y_3(4-y_3))\Delta x = 3.959909617 \end{array}$$

(4) This is a separable equation, find the general solution and determine the constant of integration from the initial condition $y(0) = 1$.

Solution: After separating the variables we have $\displaystyle\int \frac{1}{y(4-y)}\,dy = \int dx$. For the integral in y we use partial fractions: $\displaystyle\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y}$. We find that $A = B = 1/4$ and thus $\displaystyle\int \frac{1}{y(4-y)}\,dy = \frac{1}{4}\ln|y| - \frac{1}{4}\ln|4-y|$. Therefore, $\displaystyle\frac{1}{4}\ln\left|\frac{y}{4-y}\right| = x + C$. Then $\displaystyle\left|\frac{y}{4-y}\right| = e^{4x}\cdot e^{4C}$ and thus $\displaystyle\frac{y}{4-y} = A\cdot e^{4x}$.
Solving for y, we obtain $\displaystyle y = \frac{4Ae^{4x}}{1+Ae^{4x}}$.
The initial condition $y(0) = 1$ gives $\displaystyle 1 = \frac{4A}{1+A}$ and thus $A = 1/3$.

(C) Find the general solutions of the following differential equations

(1) $\displaystyle y' = \frac{y}{x(x+1)}$

Solution: This is a separable differential equation.
We have $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x(x+1)}$. For the integral on the right we use partial fractions. $\displaystyle\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.
Thus $\displaystyle\int \frac{1}{x(x+1)}\,dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C$.
We have $\displaystyle\ln|y| = \ln\left|\frac{x}{x+1}\right| + C$ and thus $\displaystyle|y| = e^{\ln\left|\frac{x}{x+1}\right|+C} = \left|\frac{x}{x+1}\right|\cdot e^C$.
Therefore $\displaystyle y = A\frac{x}{x+1}$ is the general solution of the given differential equation.

(2) $\displaystyle y' = \frac{\sqrt{1-x^2}}{e^{2y}}$.

Solution: This is a separable differential equation.
We have $\displaystyle\int e^{2y}\,dy = \int \sqrt{1-x^2}\,dx$. For the integral on the right we use the trigonometric substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$. Thus $\displaystyle\int \sqrt{1-x^2}\,dx =$

$$\int \sqrt{1-\sin\theta}\cos\theta\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C =$$

$$\frac{1}{2}\theta + \frac{1}{4}2\sin\theta\cos\theta + C = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$$

Therefore $\frac{1}{2}e^{2y} = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$ or $e^{2y} = \arcsin x + x\sqrt{1-x^2} + D$ and we have that $y = \frac{1}{2}\ln(\arcsin x + x\sqrt{1-x^2} + D)$ is the general solution to the given differential equation.

(D) Newton's Law of Cooling states that the rate at which the temperature of an object changes is proportional to the difference between the object's temperature and the surrounding temperature. A hot cup of tea with temperature 100°C is placed on a counter in a room maintained at constant temperature 20°C. Ten minutes later the tea has cooled to 76°C. How long will it take to cool off to 45°C? (Express Newton's Law as a differential equation, solve it for the temperature function, then use that to answer the question.)

Solution: Let $T(t)$ denote the temperature of the cup at time t measured in minutes from the time it was placed on the counter. The differential equation modeling this scenario is $\frac{dT}{dt} = k(T-20)$. In fact, this is an initial value problem: $T(0) = 100$ and we have the additional information $T(10) = 76$. This will help us find the constant of proportionality k. The differential equation is separable and we have $\int \frac{dT}{T-20} = \int k\,dt$. Integrating both sides we obtain $\ln|T-20| = kt+C$ and thus $T - 20 = Ae^{kt}$. Therefore $T(t) = 20 + Ae^{kt}$. Since $T(0) = 100$, we have $A = 80$. Since $T(10) = 76$, we have $76 = 20 + 80e^{10k}$. Thus $k = \frac{1}{10}\ln\frac{56}{80}$ and $T(t) = 20 + 80e^{1/10\ln(7/10)t}$. To find the time when the tea has cooled to $45°C$, we sove $20 + 80e^{1/10\ln(7/10)t} = 45$. Thus $e^{1/10\ln(7/10)t} = 25/80 = 5/16$ and the tea will be at $45°C$ after $t = 10\frac{\ln(5/16)}{\ln(7/10)} \approx 32.6$ minutes.

II. (A) Does the sequence $a_n = n\ln(1+n)$ converge? Why or why not? Does the infinite series $\sum_{n=1}^{\infty} n\ln(1+n)$ converge? Why or why not?

Solution: The sequence $a_n = n\ln(1+n)$ is not bounded and thus it does not converge. Since $\lim_{n\to\infty} n\ln(1+n) \neq 0$, the series $\sum_{n=1}^{\infty} n\ln(1+n)$ diverges (by the Divergence Test).

(B) Use the Integral Test to determine whether or not

$$\sum_{k=1}^{\infty} \frac{k}{e^k}$$

converges.

Solution: The function $f(x) = \dfrac{x}{e^x}$ is continuous and positive. Since $f'(x) = \dfrac{e^x - xe^x}{e^{2x}} = \dfrac{e^x(1-x)}{e^{2x}} < 0$ for $x > 1$, f(x) is also decreasing for $x > 1$.

Consider $\displaystyle\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx$. Using integration by parts, $u = x$, $du = dx$, $dv = e^{-x}dx$, $v = -e^{-x}$, the improper integral equals $\displaystyle\lim_{b\to\infty}\left(-be^{-b} + e^{-1} + \int_1^b e^{-x}\,dx\right) = \lim_{b\to\infty}\left(-be^{-b} + e^{-1} - e^{-b} + e^{-1}\right)$. Since $\displaystyle\lim_{b\to\infty} e^{-b} = 0$ and $\displaystyle\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty}\frac{b}{e^b} \overset{l'H}{=} \lim_{b\to\infty}\frac{1}{e^b} = 0$, the improper integral converges to $2e^{-1}$. By the Integral Test, the series $\displaystyle\sum_{k=1}^{\infty}\frac{k}{e^k}$ converges.

(C) Use the Ratio Test to determine whether or not

$$\sum_{k=0}^{\infty} \frac{3^n}{n!}$$

converges.

Solution:

$$\lim_{n\to\infty}\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1.$$

By the Ratio Test, the series converges.

(D) Determine (with justification!) whether or not the following series converge:

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}, \qquad \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{\pi^{2n}}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^{1.01}}.$$

Solution: The series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ is the p-series with $p = 1/2$ and thus it diverges.

The series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{\pi^{2n}}$ is the geometric series with ratio $\dfrac{-3}{\pi^2}$. Since the ratio is less than 1 in absolute value, the series converges. (The sum of the series is $\dfrac{1}{1+\frac{3}{\pi^2}}$.)

The series $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ is the p-series with $p = 1.01$. Since $p > 1$, the p-series converges.

(E) Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Find the 4th degree Taylor polynomial of f centered at $a = 0$. Find a factorial expression for the general term of the Taylor series.

Solution: We have $f(x) = (1+x)^{1/2}$, $f'(x) = \frac{1}{2}(x+1)^{-1/2}$, $f''(x) = -\frac{1}{2^2}(x+1)^{-3/2}$, $f'''(x) = \frac{1\cdot 3}{2^3}(x+1)^{-5/2}$, $f^{(4)}(x) = -\frac{1\cdot 3\cdot 5}{2^4}(x+1)^{-7/2}$. Thus $f(0) = 1, f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{2^2}, f'''(0) = \frac{1\cdot 3}{2^3}, f^{(4)}(0) = -\frac{1\cdot 3\cdot 5}{2^4}$. The 4th degree Taylor polynomial of f centered at $a = 0$ is $T_4(x) = 1 + \frac{1}{2}x - \frac{1}{2^2\cdot 2!}x^2 + \frac{1\cdot 3}{2^3\cdot 3!}x^3 - \frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^4$.

The general term of the Taylor series is $(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2(n-1)-1)}{2^n\cdot n!}x^n$.
(The numerator is the product of the first $n-1$ odd numbers).

(F) Consider the geometric series $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

(1) Use series manipulations to find the Taylor series of $xf'(x)$. **Solution:** We obtain the Taylor series of $f'(x)$ by differentiating the Taylor series of f term by term. $f'(x) = \sum_{k=1}^{\infty} kx^{k-1}$. To obtain the Taylor series of $xf'(x)$ we multiply each term of the Taylor series of $f'(x)$ by x. Thus $xf'(x) = \sum_{k=1}^{\infty} kx^k$.

(2) Use series manipulations to find the Taylor series of $-\ln(1-x)$.

Solution: Since $(-\ln(1-x))' = \frac{1}{1-x}$, we integrate the geometric series term by term to obtain the Taylor series for $-\ln(1-x)$.
Thus $-\ln(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k}$.

(3) Find the radius of convergence of the series in part (2), and investigate convergence at the endpoints.

Solution: Since we obtained the Taylor series by integrating the geometric series (which has radius of convergence 1), the radius of convergence of the Taylor series of $-\ln(1-x)$ is 1. This can also be found using the Ratio Test. When $x = 1$, the Taylor series of $-\ln(1-x)$ is the harmonic series and thus it diverges. When $x = -1$, we obtain the alternating harmonic series which converges. Thus the interval of converges for the Taylor series of $-\ln(1-x)$ is $[-1, 1)$.

(4) Use parts (1) and (2) to *evaluate* the sums of the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$.

Solution: The series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{k}$ is the series in (2) with $x = 1/2$. Thus its sum is $-\ln(1-1/2) = \ln(1/2) = \ln 2$.

The series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{k=1}^{\infty} k(1/2)^k$ is the series of (1) with $x = 1/2$. Then its sum is $1/2f'(1/2)$. Since $f'(x) = \dfrac{1}{(1-x)^2}$, $f'(1/2) = 4$ and the sum of the series is 2.

(G) For each of the given power series, find the interval of convergence.

$$f(x) = \sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}, \qquad g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \cdot 3^n}.$$

(In particular, give the radius of convergence, and investigate convergence at the endpoints.)

Solution: For $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}$, consider the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{\sqrt{n+1}}}{\frac{(2x)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} 2|x| \frac{\sqrt{n}}{\sqrt{n+1}} = 2|x|.$$

The series converges if $|x| < 1/2$ and it diverges if $|x| > 1/2$. Since the series is centered at 0 the radius of convergence is 1/2.

If $x = 1/2$, the series equals $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is the p-series with $p = 1/2$. Since $p < 1$, the series diverges.

If $x = -1/2$, the series equals $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Since the sequence $\dfrac{1}{\sqrt{n}}$ is decreasing and it converges to 0 as $b \to \infty$, the series converges by the Alternating Series Test.

The interval of convergence for the first series is $[1/2, 1/2)$.

We consider the Ratio Test for $g(x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \cdot 3^n}$.

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{n+1}}{(n+1)3^{n+1}}}{\frac{|x-5|^n}{n \cdot 3^n}} = \lim_{n\to\infty} \frac{|x-5| \cdot n}{3(n+1)} = \frac{|x-5|}{3}.$$

The series converges if $|x-5| < 3$ and it diverges if $|x-5| > 3$. Thus the radius of convergence is 3.

If $x - 5 = 3$, *i.e.*, $x = 8$, the series becomes the alternating harmonic series and it converges.

If $x - 5 = -3$, *i.e.*, $x = 2$, the series equals $g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ which is the negative of the harmonic series and thus it diverges.

The interval of convergence for the second series is $(2, 8]$.

(H) The second degree Taylor polynomial of $f(x)$ at $a = 0$ is $p_2(x) = c + bx + ax^2$. What can you say about the signs of a, b, c if you know $f(x)$ is increasing and concave down at $x = 0$?

Solution: Since $p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$, we have $a = f(0)$, $b = f'(0)$ and $c = \frac{f''(0)}{2}$. Since f is increasing at 0, $b > 0$. Since f is concave down at 0, $c < 0$. We do not have enough information to determine the sign of a.