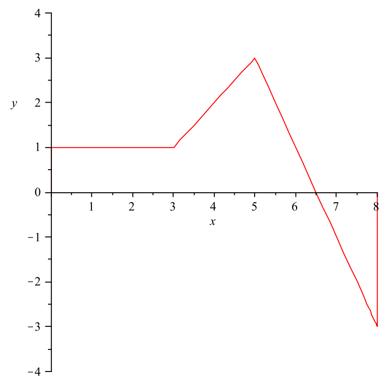
Mathematics 132 – Calculus for Physical and Life Sciences 2
Exam 1 – Review Sheet
February 18, 2008

Sample Exam Questions- Solutions I. Let

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 3 \\ x - 2 & \text{if } 3 \le x \le 5 \\ 13 - 2x & \text{if } 5 \le x \le 8 \end{cases}$$

(A) Sketch the graph $y = f(x)$.



In the rest of the parts, $F(x) = \int_0^x f(t)\, dt$, where f is the function from part A.

(B) Assuming $F(0) = 0$, Compute $F(1), F(2), F(3), F(4), F(5), F(6), F(7), F(8)$ given the information in the graph of f.

Using the area interpretation of the definite integral we have

$$F(1) = \int_0^1 f(x)\, dx = 1$$

$$F(2) = \int_0^2 f(x)\, dx = 2$$

$$F(3) = \int_0^3 f(x)\, dx = 3$$

$$F(4) = \int_0^3 f(x)\, dx + \int_3^4 f(x)\, dx = 3 + \frac{3}{2} = \frac{9}{2}$$

$$F(5) = \int_0^4 f(x)\, dx + \int_4^5 f(x)\, dx = \frac{9}{2} + \frac{5}{2} = 7$$

$$F(6) = \int_0^5 f(x)\, dx + \int_5^6 f(x)\, dx = 7 + 2 = 9$$

$$F(7) = \int_0^6 f(x)\, dx + \int_6^{13/2} f(x)\, dx + \int_{13/2} 7 = 9 + \frac{1}{4} - \frac{1}{4} = 9$$

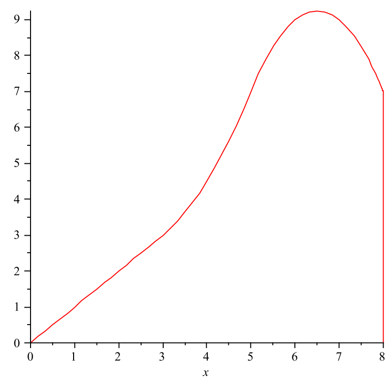
$$F(8) = \int_0^5 f(x)\,dx + \int_5^{13/2} f(x)\,dx + \int_{13/2} 8f(x)\,dx = \int_0^5 f(x)\,dx = 7$$ (the last two integrals cancel since they represent equal areas with opposite signs).

(C) Are there any critical points of F? If so, find them and say whether they are local maxima, local minima, or neither. If not, say why not.

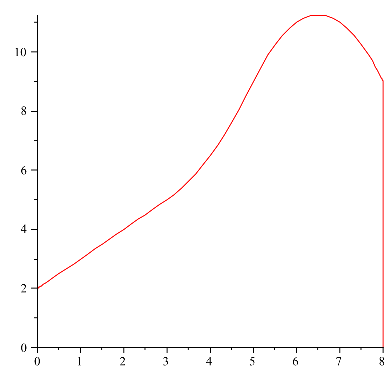
By the Fundamental Theorem of Calculus, $F'(x) = f(x)$. Since $f(13/2) = 0$, the point $x = 13/2$ is a critical point. Since $F' = f$ changes sign from positive to negative at the critical point, $x = 13/2$ is a local maximum.

(D) Sketch the graph $y = F(x)$ if $F(0) = 0$, and also if $F(0) = 2$.

If $F(0) = 0$ we have



If $F(0) = 2$ we have



Ignore the vertical lines in the graph.

II. Find the derivatives of the following functions (A) $f(x) = \int_0^x \sin(t)/t\,dt$.

$$f'(x) = \frac{\sin x}{x}$$

(B) $g(x) = \int_5^{x^3} \tan^4(t)\ dt$.

$g(x) = m(x^3)$, where $m(x) = \displaystyle\int_5^x \tan^4(t)\ dt$. Then, $g'(x) = m'(x^3) \cdot 3x^2 = \tan^4(x^3) \cdot 3x^2$.

(C)) $h(x) = \int_{-3x}^{5x} e^{t^2} \sin(t)\ dt$.

$h(x) = n(x) + l(x)$, where $n(x) = \displaystyle\int_{-3x}^{0} e^{t^2} \sin(t)\ dt$ and $l(x) = \displaystyle\int_0^{5x} e^{t^2} \sin(t)\ dt$. Then $h'(x) = n'(x) + l'(x) = -(e^{(-3x)^2} \sin(-3x)) \cdot (-3) + 5 \cdot e^{(5x)^2} \sin(5x) = 3e^{9x^2} \sin(-3x) + 5e^{25x^2} \sin(5x)$.

III.

(A) Compute $\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\ dx$

$$\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\ dx = x^5 - 2x^{3/2} + e^x + 2\ln|x| + C$$

(B) Apply a u-substitution to compute $\int x(4x^2 - 3)^{3/5}\ dx$

$u = 4x^2 - 3$, $du = 8x\,dx$. Then $\displaystyle\int x(4x^2-3)^{3/5}\ dx = \int \frac{1}{8} u^{3/5}\,dx = \frac{1}{8}\frac{u^{8/5}}{8/5} + C = \frac{5}{64}(4x^2 - 3)^{8/5} + C$

(C) Apply a u-substitution to compute $\int_1^2 e^{\sin(\pi x)} \cos(\pi x)\ dx$

$u = \sin(\pi x)$, $du = \pi \cos(\pi x)$. Then $\displaystyle\int_1^2 e^{\sin(\pi x)} \cos(\pi x)\ dx = \frac{1}{\pi} \int_0^0 e^u\,du = 0$

(D) Do you need partial fractions to compute

$$\int \frac{t^2 + 1}{t^3 + 3t + 3}\ dt?$$

Explain, and give a simpler method.

No. Just do a u-substitution. $u = t^3 + 3t + 3$, $du = (3t^2 + 3)dt = 3(t^2 + 1)dt$. Then $\displaystyle\int \frac{t^2+1}{t^3+3t+3}\ dt = \frac{1}{3} \int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|t^3 - 3t + 3| + C$.

(E) Apply integration by parts to compute $\int x^2 e^{-2x}\ dx$

$u = x^2 \qquad du = 2x\,dx$
$dv = e^{-2x} \qquad v = -\frac{1}{2}e^{-2x}$
Then $\displaystyle\int x^2 e^{-2x}\ dx = -\frac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\,dx$. Integration by parts again.
$u = x \qquad du = dx$
$dv = e^{-2x} \qquad v = -\frac{1}{2}e^{-2x}$

Then $\int x^2e^{-2x}\,dx = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

(F) Apply partial fraction decomposition to compute

$$\int \frac{1}{x(x-1)(x+2)}\,dx$$

$\frac{1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$. Then $A(x-1)(x+2) + Bx(x+2) + Cx(x-1) = 1$. If $x = 0$, we get $-2A = 1$. Thus, $A = -\frac{1}{2}$. If $x = 1$, we get $3B = 1$. Thus $B = \frac{1}{3}$. If $x = -2$, we get $6C = 1$. Thus $C = \frac{1}{6}$. Therefore

$\int \frac{1}{x(x-1)(x+2)}\,dx = -\frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx + \frac{1}{6}\int \frac{1}{x+2}\,dx =$

$-\frac{1}{2}\ln|x| + \frac{1}{3}\ln|x-1| + \frac{1}{6}\ln|x+2| + C.$

(G) Which trigonometric substitution would you apply to compute $\int \frac{1}{u\sqrt{a^2-u^2}}\,du$? What trigonometric integral do you get after making the substitution? Complete the derivation of the integral.

Use the trigonometric substitution $u = a\sin\theta,\ du = a\cos\theta\,d\theta$. Then

$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = \int \frac{1}{a\sin\theta\sqrt{a^2-a^2\sin^2\theta}} a\cos\theta\,d\theta = \int \frac{\cos\theta}{\sin\theta a\sqrt{\cos^2\theta}}\,d\theta =$

$\frac{1}{a}\int \frac{\cos\theta}{\sin\theta\cos\theta}\,d\theta = \frac{1}{a}\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{a}\int \csc\theta\,d\theta$. Now we use the table of integrals (formula 15) to obtain that the integral equals $\frac{1}{a}\ln|\csc\theta - \cot\theta| + C$. If $u = a\sin\theta$, then $\csc\theta = a/u$ and $\cot\theta = \frac{\sqrt{a^2-u^2}}{u}$. Therefore

$$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = \frac{1}{a}\ln|\frac{a}{u} - \frac{\sqrt{a^2-u^2}}{u}| + C.$$

IV. Compute each of the integrals below using some combination of basic rules, substitution, integration by parts, the table of integrals, partial fractions, and trigonometric substition. You must show all work for full credit.

(A)

$$\int \frac{x^3}{x^2+4x}\,dx$$

First simplify and do long division to get $\frac{x^3}{x^2+4x} = \frac{x^2}{x+4} = x - 4 + \frac{16}{x+4}$. Therefore

$\int \frac{x^3}{x^2+4x}\,dx = \int (x-4)\,dx + \int \frac{16}{x+4}\,dx = \frac{x^2}{2} - 4x + 16\ln|x+4| + C.$

(B)

$$\int (x^2+2x)^{3/2}\ dx$$

First complete the square to get $\displaystyle\int (x^2+2x)^{3/2}\ dx = \int ((x+1)^2-1)^{3/2}\, dx$. Now do the substitution $u = x+1,\ du = dx$. The integral equals $\int (u^2-1)^{3/2}\, du$. We can split this up as

$$\begin{aligned}\int (u^2-1)^{3/2}\, du &= \int (u^2-1)\sqrt{u^2-1}\ du \\ &= \int u^2\sqrt{u^2-1}\ du - \int \sqrt{u^2-1}\ du\end{aligned}$$

The first integral is #40 in the table, and the second is # 39, so we obtain

$$\begin{aligned}&= \frac{u}{8}(2u^2-1)\sqrt{u^2-1} - \frac{1}{8}\ln|u+\sqrt{u^2-1}| - \frac{u}{2}\sqrt{u^2-1} + \frac{1}{2}\ln|u+\sqrt{u^2-1}| + C \\ &= \frac{1}{4}(x+1)^3\sqrt{x^2+2x} - \frac{5}{8}(x+1)\sqrt{x^2+2x} + \frac{3}{8}\ln|x+1+\sqrt{x^2+2x}| + C\end{aligned}$$

(C)

$$\int \frac{e^{\sqrt{\sin(x)}}\cos(x)}{\sqrt{\sin(x)}}\ dx$$

Use the substitution $u = \sqrt{\sin x}, du = \dfrac{1}{2\sqrt{\sin x}}\cos x\, dx$. Then

$$\int \frac{e^{\sqrt{\sin(x)}}\cos(x)}{\sqrt{\sin(x)}}\ dx = 2\int e^u\, du = 2e^u + C = 2e^{\sqrt{\sin x}} + C.$$

(D) Note: There was a typo in the statement of this one. The intended integral was

$$\int \frac{dz}{(z+3)^2\sqrt{z^2+6z+10}}$$

Completing the square we obtain

$$\int \frac{dz}{(z+3)^2\sqrt{z^2+6z+10}} = \int \frac{1}{(z+3)^2\sqrt{(z+3)^2+1}}\, dz.$$

Now we perform a u-substitution: $u = z+3,\ du = dz$ and the integral becomes

$$\int \frac{1}{u^2\sqrt{u^2+1}}\, du$$

which is in the form of #28 in the table. The answer is

$$-\frac{\sqrt{u^2+1}}{u} + C = -\frac{\sqrt{z^2+6z+10}}{z+3} + C.$$