Mathematics 132 – Calculus for Physical and Life Sciences, 2
Summary of Trigonometric Substitutions
February 14, 2005

The trigonometric substitution method handles many integrals containing expressions like

$$\sqrt{a^2 - x^2},\ \sqrt{x^2 + a^2},\ \sqrt{x^2 - a^2}$$

(possibly including expressions without the square roots!) The basis for this approach is the trigonometric identities

$$1 = \sin^2\theta + \cos^2\theta$$
$$\Rightarrow \sec^2\theta = \tan^2\theta + 1.$$

from which we derive other related identities:

$$\sqrt{a^2 - (a\sin\theta)^2} = a\cos\theta$$
$$\sqrt{(a\tan\theta)^2 + a^2} = a\sec\theta$$
$$\sqrt{(a\sec\theta)^2 - a^2} = a\tan\theta$$

Hence,

1. If our integral contains $\sqrt{a^2 - x^2}$, the substitution $x = a\sin\theta$ will convert this radical to the simpler form $a\cos\theta$.
2. If our integral contains $\sqrt{x^2 + a^2}$, the substitution $x = a\tan\theta$ will convert this radical to the simpler form $a\sec\theta$.
3. If our integral contains $\sqrt{x^2 - a^2}$, the substitution $x = a\sec\theta$ will convert this radical to the simpler form $a\tan\theta$.

Then we substitute for the rest of the integral, convert to $\sin\theta, \cos\theta$ using $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$, apply the entries in section IV of the table of integrals, and convert back to the original variable.

Two Examples

A) Compute $\int \frac{u^2}{\sqrt{a^2-u^2}}\,du$. Solution: The $\sqrt{a^2 - u^2}$ tells us that we want the sine substitution: $u = a\sin\theta$. Then $du = a\cos\theta\ d\theta$, and the integral becomes:

$$\int \frac{a^3\sin^2\theta\cos\theta\ d\theta}{a\cos\theta} = a^2\int \sin^2\theta\ d\theta$$

We apply the reduction formula # 17 (with n = 2) in the table to this:

$$= \frac{-a^2}{2}\sin\theta\cos\theta + \frac{a^2}{2}\int\ d\theta = \frac{-a^2}{2}\sin\theta\cos\theta + \frac{a^2\theta}{2} + C.$$

Then, we convert back to functions of u using the substitution equation $u = a\sin\theta$. From this,

$$\theta = \arcsin(u/a), \qquad \cos\theta = \sqrt{a^2 - u^2}/a, \qquad \sin\theta = u/a$$

so the integral is

$$\int \frac{u^2}{\sqrt{a^2 - u^2}}\, du = \frac{-1}{2} u\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin(u/a) + C.$$

B) Compute $\int \frac{dx}{x\sqrt{x^2+16}}$. Solution: The $\sqrt{x^2+16}$ indicates that we want the tangent substitution $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\; d\theta$ and the integral becomes:

$$\int \frac{4\sec^2\theta\; d\theta}{4\tan\theta \cdot 4\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\, d\theta = \frac{1}{4}\int \frac{1}{\sin\theta}\, d\theta$$

From # 20 in the table of integrals,

$$\frac{1}{4}\int \frac{1}{\sin\theta}\, d\theta = \frac{1}{8}\ln\left|\frac{\cos(\theta) - 1}{\cos(\theta) + 1}\right| + C$$

Then, from $x = 4\tan\theta$, we get $\cos\theta = \frac{4}{\sqrt{x^2+16}}$ and the integral equals:

$$\frac{1}{8}\ln\left|\frac{\frac{4}{\sqrt{x^2+16}} - 1}{\frac{4}{\sqrt{x^2+16}} + 1}\right| + C$$

Multiplying top and bottom by $\sqrt{x^2+16}$ inside the logarithm, we get:

$$\int \frac{dx}{x\sqrt{x^2+16}} = \frac{1}{8}\ln\left|\frac{4 - \sqrt{x^2+16}}{4 + \sqrt{x^2+16}}\right| + C$$