College of the Holy Cross, Spring Semester, 2005
Math 132 Answers to Review Sheet for Midterm 3

1. The graph at right shows either a pdf or a cdf. Which type of function is it, and why? If it is a pdf, sketch the graph of the corresponding cdf; if it is a cdf, sketch the graph of the corresponding pdf.

   Solution The graph encloses more than one unit of area (and is non-decreasing and seems to approach 1 as $x \to \infty$), so it's a cdf. The corresponding density has one bump, centered just to the right of $x = 0$.

2. Let x be the size of an email message in kilobytes (KB). An ISP finds that the fraction of emails between x and $x + \Delta x$ KB in size is about $cxe^{-0.01x}\Delta x$ (that is, the density function for x is $p(x) = cxe^{-0.01x}$).

   (a) $c = (0.01)^2 = 10^{-4}$

   (b) Fraction of email messages that are at most 100 KB in size: $1 - \frac{2}{e} \simeq 0.264$

   (c) Fraction that are at least 50 KB in size: $\frac{3}{2\sqrt{e}} \simeq 0.91$.

   (d) Median size of an email message: About 167.8 KB.

3. In a certain population, the height x of an individual in inches is normally distributed, with a mean of 68 and a standard deviation of 4. In other words, heights are modeled by the density function $p(x) = \frac{1}{4\sqrt{2\pi}}e^{-(x-68)^2/32}$.

   (a) Write down a definite integral whose value gives the fraction of the population whose height is between 68 and 72 inches, and perform the change of variables $z = (x - 68)/4$ on your integral.

   Solution $\dfrac{1}{4\sqrt{2\pi}}\displaystyle\int_{68}^{72} e^{-(x-68)^2/32}\,dx = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^1 e^{-z^2/2}\,dz.$

   (b) Write out the Taylor series for e^u giving both the first four non-zero terms and the summation form.

   Solution $e^u = \displaystyle\sum_{k=0}^{\infty} \frac{u^k}{k!} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$

   (c) Use your answer from (b) to find the Taylor series for

$$\int_0^x e^{-z^2/2}\,dz,$$

   giving both the first four non-zero terms and the summation form.

Solution $\int_0^x e^{-z^2/2}\,dz = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)\cdot 2^k \cdot k!} = x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 2^2\cdot 2!} - \frac{x^7}{7\cdot 2^3\cdot 3!} + \cdots$

(d) Estimate the integral in part (a) using the four-term series from part (c).

Solution $\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-z^2/2}\,dz \simeq \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{3} \frac{(-1)^k}{(2k+1)\cdot 2^k\cdot k!} = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336}\right) \simeq 0.34$

(The absolute error of this estimate is at most $\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{9\cdot 2^4\cdot 4!} \simeq 0.0001$

4. (a) Use the Comparison Test to determine whether or not $\sum_{n=0}^{\infty} \frac{n+3^n}{2^n}$ converges.

Solution Diverges by comparison with $\sum_{n=0}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$.

(b) Use the Integral Test to determine whether or not $\sum_{k=0}^{\infty} \frac{k}{e^k}$ converges.

Solution Converges (see also Question 2 above.)

(c) Use the Ratio Test to determine whether or not $\sum_{k=0}^{\infty} \frac{3^n}{n!}$ converges.

Solution Converges, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3}{n+1}\right| = 0 < 1$.

5. Determine (with justification!) whether or not the following series converge:

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}, \qquad \sum_{k=3}^{\infty} (-1)^{k-1}\frac{1}{\ln k}, \qquad \sum_{n=10}^{\infty} \frac{n^2+n}{3n^3-1000}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{1.01}}.$$

Solution The first and third diverge: Use the integral test for the first, comparison with the harmonic series for the third. The second and fourth converge: The second is alternating, and the terms decrease to 0 in absolute value, while the fourth converges by the integral test.

6. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Find the 4th degree Taylor polynomial of f centered at $a = 0$. Find a factorial expression for the general term of the Taylor series.

Solution $p_4(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{15}{16 \cdot 24}x^4$. The first few derivatives are $f'(0) = \frac{1}{2}$,

$f^{(2)}(0) = -\frac{1}{2} \cdot \frac{1}{2}$ $\quad f^{(3)}(0) = \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ $\quad f^{(4)}(0) = -\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ $\quad f^{(5)}(0) = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$.

For $k \geq 2$, the general expression is

$$f^{(k)}(0) = (-1)^{k-1} \frac{(2k-3)}{2} \cdot \frac{(2k-5)}{2} \cdots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = (-1)^{k-1} \frac{(2k-2)!}{2^k} \cdot \frac{1}{2^{k-1}(k-1)!}$$

We do not expect you to construct such formulas, and certainly not under test pressure.

7. Consider the geometric series $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

(a) Differentiate and multiply by x; the Taylor series of $xf'(x) = \frac{x}{(1-x)^2}$ is $\sum_{k=0}^{\infty} kx^k$.

(b) Integrate; the Taylor series of $-\ln(1-x)$ is $\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k}$.

(c) Find the radius of convergence of the series in part (b), and investigate convergence at the endpoints.

Solution The radius is 1, the interval of convergence is $-1 \leq x < 1$.

(d) Set $x = 1/2$ to see that $\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} = \ln 2$, $\sum_{k=1}^{\infty} \frac{k}{2^k} = 2$.

8. For each of the given power series, find the interval of convergence.

$$f(x) = \sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}, \qquad g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \cdot 3^n}.$$

(In particular, give the radius of convergence, and investigate convergence at the endpoints.)

Solution For $f(x)$, $-\frac{1}{2} \leq x < \frac{1}{2}$. For $g(x)$, $2 < x \leq 8$. (The first has radius $1/2$, the second has radius 3.)

9. The second degree Taylor polynomial of $f(x)$ at $a = 0$ is $p_2(x) = C_0 + C_1x + C_2x^2$. What can you say about the signs of C_0, C_1, and C_2 if you know the graph of $f(x)$ is:

Solution $C_0 < 0$, C_1 and $C_2 > 0$.

10. Use the error bound for Taylor approximations to estimate the number of decimal places of accuracy if the 6th degree Taylor polynomial at $a = 0$ is used to approximate $\cos(0.8)$. Do the same for the nth degree polynomial in general. What happens to the error bound as $n \to \infty$?

Solution With a polynomial of degree $2n$, the "standard" error bound is

$$|E_{2n}(0.8)| \le \frac{(0.8)^{2n+2}}{(2n+2)!},$$

the absolute value of the first term of the Taylor series that is omitted from the approximating polynomial. (This observation rests on the fact that the series for $\cos(0.8)$ alternates and the terms decrease in absolute value.)

In particular, the degree-6 polynomial estimate is accurate to $\pm 4.16 \times 10^{-6}$, or 5 decimal places. The error bounds decrease rapidly to 0 as $n \to \infty$: The numerators decrease exponentially and the denominators grow "super-exponentially" (asymptotically faster than any exponential function).