Mathematics 132 – Calculus for Physical and Life Sciences, 2
Summary of Partial Fractions
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Every rational function (quotient of polynomials) can be written as *a polynomial plus a sum of one or more terms of the following forms*:

$$\frac{Ax+B}{(x^2+2bx+c)^k}, \qquad \frac{C}{(x-a)^k}.$$

A rational function expressed this way is said to be *decomposed into partial fractions.* The process of finding this decomposition is as follows: Given a rational function $\frac{f(x)}{g(x)}$,

1. First, if the degree of g is larger already, just proceed to step 2 with $\frac{f(x)}{g(x)}$. Otherwise, if the degree of $f(x)$ is greater than or equal to the degree of $g(x)$, *divide* $g(x)$ into $f(x)$ using polynomial long division and write $f(x) = q(x)g(x) + r(x)$ for some quotient $q(x)$ and remainder $r(x)$. Then

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

   and the degree of $r(x)$ is less than the degree of $g(x)$. Continue to step 2 with $\frac{r(x)}{g(x)}$.
2. *Factor* $g(x)$ completely into a product of powers of linear polynomials and powers of irreducible quadratic polynomials. (The fact that this can always be done is one form of the so-called "Fundamental Theorem of Algebra". The famous mathematician and physicist Carl Friedrich Gauss gave the first complete proof of this result in 1799.)
3. Assemble the partial fractions: For each $(x-a)^m$ appearing in the factorization of $g(x)$, include a *group of terms*

$$\frac{C_1}{(x-a)} + \frac{C_2}{(x-a)^2} + \cdots + \frac{C_m}{(x-a)^m}$$

   For each $(x^2+2bx+c)^n$ appearing in the factorization of $g(x)$, include a *group of terms*:

$$\frac{A_1x+B_1}{x^2+2bx+c} + \frac{A_2x+B_2}{(x^2+2bx+c)^2} + \cdots + \frac{A_nx+B_n}{(x^2+2bx+c)^n}$$

4. Set the rational function from step 1 (either the original f/g or r/g as appropriate) equal to the sum of the partial fractions, clear denominators, and solve for the coefficients. This last step can be done either by substituting well-chosen x-values, or by equating coefficients of like powers of x on both sides and solving the resulting system of equations.

An interesting consequence of the partial fraction method is that it shows, in principle, that

Theorem. *Every rational function has elementary antiderivatives.*

We add the "in principle" here because determining the factorization of $g(x)$ may not be simple in some cases, and we have not yet seen the methods that allow us to integrate the partial fractions with higher powers of quadratic polynomials in the denominator.

In any case, here's an example showing most of the different things that can happen. Say we want to decompose

$$\frac{x^4+1}{x^4+x^2}$$

into partial fractions. First, the degree of the top is 4 and the degree of the bottom is 4 also, so we begin by dividing the bottom into the top:

$$x^4+1 = 1(x^4+x^2) + (-x^2+1)$$

This means

$$\frac{x^4+1}{x^4+x^2} = 1 + \frac{-x^2+1}{x^4+x^2}$$

We continue to step 2 with the rational function

$$\frac{-x^2+1}{x^4+x^2}$$

The factorization of the denominator is

$$x^4+x^2 = x^2(x^2+1)$$

Hence our general recipe for assembling the partial fractions says:

$$\frac{-x^2+1}{x^4+x^2} = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{Ax+B}{x^2+1}$$

So after clearing denominators and collecting terms,

$$-x^2+1 = C_1x(x^2+1) + C_2(x^2+1) + (Ax+B)x^2 = (A+C_1)x^3 + (B+C_2)x^2 + C_1x + C_2$$

Equating coefficients of like powers of x,

$$C_2 = 1, C_1 = 0, B + C_2 = -1, A + C_1 = 0,$$

so

$$C_2 = 1, C_1 = 0, B = -2, A = 0.$$

In other words,

$$\frac{x^4+1}{x^4+x^2} = 1 + \frac{-x^2+1}{x^4+x^2} = 1 + \frac{1}{x^2} + \frac{-2}{x^2+1}$$

From this we can integrate:

$$\int \frac{x^4+1}{x^4+x^2}\, dx = \int 1 + \frac{1}{x^2} + \frac{-2}{x^2+1}\, dx = x + \frac{-1}{x} - 2\arctan(x) + C$$