Math 131, Midterm 3 Solutions

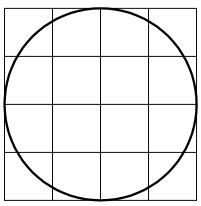
1. [5 points each] Match each parametric curve with its graph. (Each graph shows the square $-1 \le x \le 1$, $-1 \le y \le 1$.)

   (a) $x(t) = -1 + 2t$, $y(t) = 1 - 2t$, $0 \le t \le 1$ ○ I ○ II ○ III ○ IV ⊗ V
   Explanation. This is a line with slope $-2/2 = -1$.

   (b) $x(t) = \cos(2t)$, $y(t) = \sin(2t)$, $0 \le t \le \pi$ ⊗ I ○ II ○ III ○ IV ○ V
   Explanation. Points (x, y) on the curve satisfy $x^2 + y^2 = 1$, so they lie on the unit circle.

   (c) $x(t) = \cos t$, $y(t) = \frac{1}{2} \sin t$, $0 \le t \le 2\pi$ ○ I ○ II ○ III ⊗ IV ○ V
   Explanation. This is the standard parametrization of the unit circle, but with the y-coordinate scaled by a factor of $1/2$, so it must be an ellipse.

   (d) $x(t) = 1 - 2t$, $y(t) = -t$, $0 \le t \le 1$ ○ I ⊗ II ○ III ○ IV ○ V
   Explanation. This is a line with slope $-1/-2 = 1/2$.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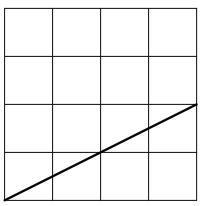
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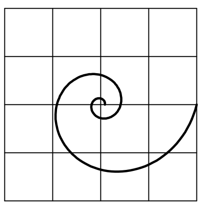
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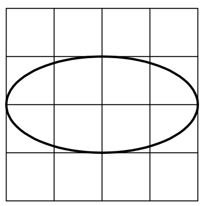
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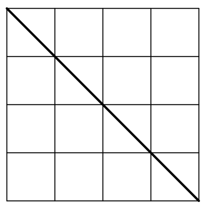
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2. [5 points each] Compute the indicated limits. Show all work for full credit.

(a) $\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{x-1}$

Answer. Since $\cos(\pi/2) = 0$, the numerator and denominator both approach zero. By L'Hopital's Rule,

$$\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{x-1} = \lim_{x\to 1} \frac{-\frac{\pi}{2}\sin(\frac{\pi}{2}x)}{1} = -\frac{\pi}{2}$$

Limit = $\boxed{-\frac{\pi}{2}}$

(b) $\lim_{x\to 0} \frac{x^2+1}{x^2+3}$

Answer.

$$\lim_{x\to 0} \frac{x^2+1}{x^2+3} = \frac{\lim_{x\to 0} x^2+1}{\lim_{x\to 0} x^2+3} = \frac{1}{3}$$

Limit = $\boxed{\frac{1}{3}}$

(c) $\lim_{x\to\infty} \frac{x\ln x}{x^{1.01}}$ **Answer.** First simplify.

$$\lim_{x\to\infty} \frac{x\ln x}{x^{1.01}} = \lim_{x\to\infty} \frac{\ln x}{x^{0.01}}$$

The numerator and denominator both approach ∞, so by L'Hopital's Rule,

$$\lim_{x\to\infty} \frac{\ln x}{x^{0.01}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{0.01x^{-0.99}} = \lim_{x\to\infty} \frac{1}{0.01x^{0.01}} = 0$$

Limit = $\boxed{0}$

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3. Each part refers to the function $f(x) = 2x^3 - 6x^2$.

(a) [10 points] Find and classify (local min/max, or neither) the critical points of f.

Answer. The derivative of f is $f'(x) = 6x^2 - 12x = 6x(x-2)$, so the critical points of f are $x = 0$ and $x = 2$.

To determine the type of each critical point, apply either the first or second derivative test.

First Derivative Test. Since $f'(-1) = 18$, $f'(1) = -6$ and $f'(3) = 18$, the sign of f' changes from $+$ to $-$ at $x = 0$, so f has a local maximum at $x = 0$, and the sign of f' changes from $-$ to $+$ at $x = 2$, so f has a local minimum at $x = 2$.

Second Derivative Test. The second derivative of f is $f''(x) = 12x - 12$. Since $f''(0) = -12$ is negative, f has a local maximum at $x = 0$, and since $f''(2) = 12$ is positive, f has a local minimum at $x = 2$.

(b) [5 points] Find the maximum and minimum values of $f(x)$ if $-\frac{3}{2} \leq x \leq \frac{5}{2}$.

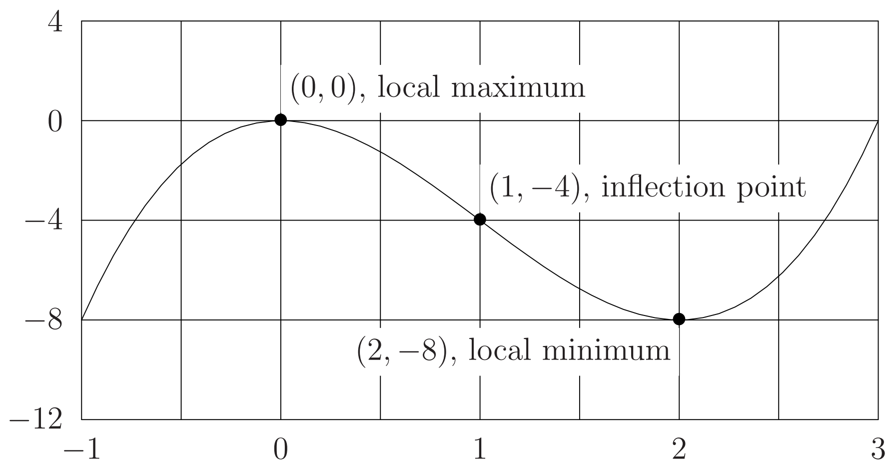
Answer. To find the global maximum and minimum, evaluate f at each critical point and at each endpoint:

$$\begin{aligned} f(0) &= 0 \\ f(2) &= -8 \\ f(-\frac{3}{2}) &= -20.25 \\ f(\frac{5}{2}) &= -6.25 \end{aligned}$$

Minimum value: -20.25 Maximum value: 0

(c) [5 points] In the grid provided, sketch the graph $y = f(x)$ for $-1 \leq x \leq 3$. (Note that axis labels are provided.) For full credit, clearly indicate the critical and inflection point(s) in this interval, and label each such point with *both coordinates*.

Answer. $f''(x) = 12x - 12 = 12(x - 1)$, so f'' changes sign at $x = 1$. Therefore f has an inflection point at $(1, -4)$. The critical points have coordinates $(0, 0)$ and $(2, -8)$.



4. [15 points] Crusader Movie Rentals finds that they can rent 160 movies per night at \$1 per movie. For every dollar that the rental fee increases, 40 fewer movies are rented. What price should be charged to maximize the revenue (total rental income)?

Answer. Let p be the price in dollars per movie, and let n be the number of movies that can be rented at that price. The revenue is then $R = np$. Let x be the increase in the price from 1 dollar. Then $p = 1 + x$, and $n = 160 - 40x$, so

$$R(x) = (1 + x)(160 - 40x) = 160 + 120x - 40x^2.$$

The domain of R is $-1 \leq x \leq 4$. Since

$$R'(x) = 120 - 80x,$$

the critical point of R is $x = \frac{3}{2} = 1.5$. Since $R(-1) = R(4) = 0$ and $R(1.5) = 250$, the revenue is maximized when $x = 1.5$, i.e. when $p = 2.5$.

Revenue maximized when $p =$ \$2.50

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5. [15 points] Find the equation of the line tangent to the curve $xy - 2x - 3y + 1 = 0$ at the point $(-2, 1)$.

Answer. Use implicit differentiation:

$$x\frac{dy}{dx} + 1 \cdot y - 2 - 3\frac{dy}{dx} = 0$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2 - y}{x - 3}$$

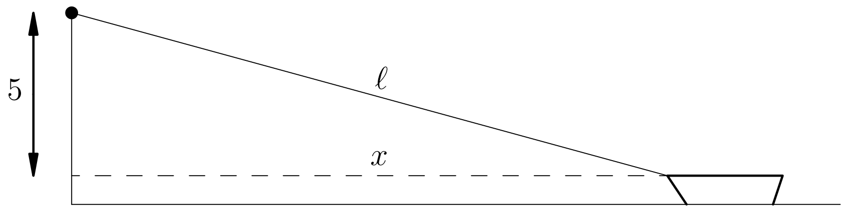
When $x = -2$ and $y = 1$,

$$\frac{dy}{dx} = \frac{1}{-5} = -\frac{1}{5}.$$

So the slope of the tangent line is $m = -\frac{1}{5}$. Applying the point-slope formula gives $y - 1 = -\frac{1}{5}(x + 2)$.

Equation of tangent line at $(-2, 1)$: $y - 1 = -\frac{1}{5}(x + 2)$ or $y = -\frac{1}{5}x + \frac{3}{5}$

6. A boat is drawn into a dock by a rope over a small pulley. The pulley is five feet higher than the bow of the boat (see figure). Let ℓ be the length of rope, x the distance from the boat to the dock.



(a) [5 points] Find an equation relating ℓ and x, and determine x when $\ell = 13$.

Answer. Use the Pythagorean Theorem.

Equation relating ℓ and x: $\boxed{x^2 + 25 = \ell^2}$

When $\ell = 13$, $x = \boxed{12}$

(b) [10 points] Suppose the rope is drawn in at 3 ft/sec. How fast is the boat moving when the length of the rope is 13 feet?

Answer. We are given $\frac{d\ell}{dt} = -3$ and want to find $\frac{dx}{dt}$ when $\ell = 13$. Differentiate the equation $x^2 + 25 = \ell^2$ with respect to t:

$$2x\frac{dx}{dt} = 2\ell\frac{d\ell}{dt}$$

When $\ell = 13$, $x = 12$, so

$$\begin{aligned} 2(12)\frac{dx}{dt} &= 2(13)(-3) \\ \frac{dx}{dt} &= -3.25 \end{aligned}$$

Speed = $\boxed{3.25 \text{ ft/sec}}$