College of the Holy Cross, Fall Semester, 2004
Math 131, Midterm 2 Solutions

1. [5 points each] Compute the following derivatives. You may use any correct method.

(a) $\dfrac{d}{dx}\Big(5x\sqrt{x} - \dfrac{2}{x^3} + 11x - 4\Big)$

Solution First rewrite $x\sqrt{x} = x^{3/2}$ and $\frac{2}{x^3} = 2x^{-3}$. Then by the power rule,

$$\frac{d}{dx}\Big(5x^{3/2} - 2x^{-3} + 11x - 4\Big) = \frac{15}{2}x^{1/2} + 6x^{-4} + 11.$$

(b) $\dfrac{d}{dt}(t^2 e^{-5t})$

Solution By the product rule,

$$\frac{d}{dt}(t^2 e^{-5t}) = t^2 \frac{d}{dt}(e^{-5t}) + e^{-5t}\frac{d}{dt}(t^2)$$

Using the chain rule on the first term, this equals

$$t^2 e^{-5t}(-5) + e^{-5t}(2t)$$

which simplifies to $(2t - 5t^2)e^{-5t}$.

(c) $\dfrac{d}{dz}8(z^2 + 4\cos z + 2)^3$

Solution By the chain rule,

$$\frac{d}{dz}8(z^2 + 4\cos z + 2)^3 = 24(z^2 + 4\cos z + 2)^2(2z - 4\sin z).$$

(d) $\dfrac{d}{dx}\left(\dfrac{x}{\sin x}\right)$

Solution By the quotient rule,

$$\frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{\sin x - x\cos x}{\sin^2 x}$$

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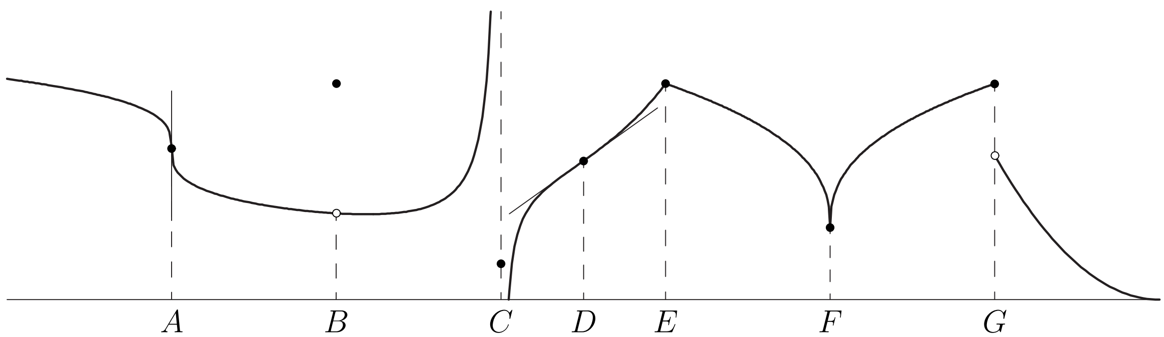
2. [15 points] The graph of a function f is shown below with several points marked. Check the appropriate boxes.



Point at which	A	B	C	D	E	F	G
f is not continuous		✓	✓				✓
f is not differentiable	✓	✓	✓		✓	✓	✓

3. [5 points each] Compute the indicated limits. Show all work for full credit.

(a) $\displaystyle\lim_{x\to 1} \frac{3x^2-5x-2}{x^2-4x+4}$

Solution Since $\displaystyle\lim_{x\to 1} x^2-4x+4 = 1 \neq 0$,

$$\lim_{x\to 1} \frac{3x^2-5x-2}{x^2-4x+4} = \frac{\displaystyle\lim_{x\to 1} 3x^2-5x-2}{\displaystyle\lim_{x\to 1} x^2-4x+4} = \frac{-4}{1} = -4$$

(b) $\displaystyle\lim_{x\to 2} \frac{3x^2-5x-2}{x^2-4x+4}$

Solution Since $\lim_{x\to 2} 3x^2-5x-2=0$ and $\lim_{x\to 2} x^2-4x+4=0$, we cannot just "plug in" to compute the limit. Factoring gives

$$\frac{3x^2-5x-2}{x^2-4x+4} = \frac{(x-2)(3x+1)}{(x-2)^2} = \frac{3x+1}{x-2} \qquad \text{for } x \neq 2$$

The resulting function $\frac{3x+1}{x-2}$ has a vertical asymptote at $x=2$ since $x=2$ is a root of the denominator and not a root of the numerator. Consequently

$$\lim_{x\to 2} \frac{3x^2-5x-2}{x^2-4x+4} = \lim_{x\to 2} \frac{3x+1}{x-2}$$

does not exist.

(c) $\displaystyle\lim_{x\to 1^-} 3\cdot\frac{x-1}{|x-1|}+1$

Solution For $x<1$, $x-1<0$ so $|x-1|=-(x-1)$ and therefore

$$3\cdot\frac{x-1}{|x-1|}+1 = 3\cdot\frac{x-1}{-(x-1)}+1 = -3+1=-2$$

so

$$\lim_{x\to 1^-} 3\cdot\frac{x-1}{|x-1|}+1=-2.$$

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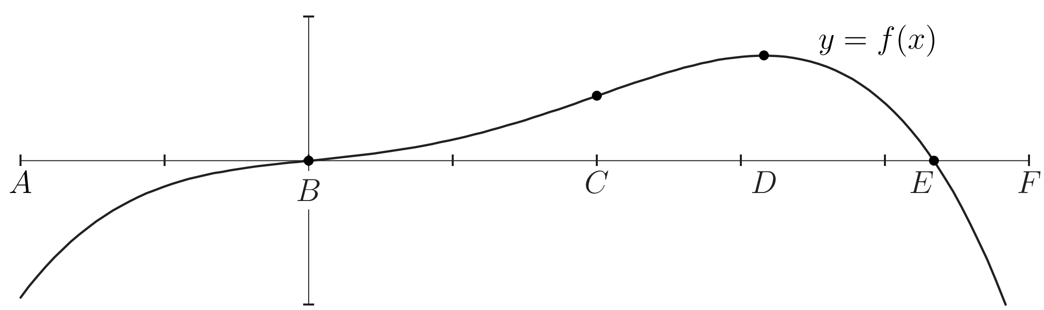
4. [5 points each] Each part refers to the graph shown.



(a) Find all intervals on which $f(x)>0$.

Solution The graph lies above the x-axis between B and E, so $f(x)>0$ for $B<x<E$.

(b) Find all intervals on which $f'(x) > 0$.

Solution The function is increasing from A to D, so $f'(x) > 0$ for $A < x < D$.

(c) Find all intervals on which $f''(x) > 0$.

Solution The graph is concave up from B to C, so $f''(x) > 0$ for $B < x < C$.

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5. (a) [5 points] State the limit definition of the derivative $f'(x)$.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(b) [10 points] Use the **definition** to compute the derivative function of $f(x) = \dfrac{1}{3x}$.

Solution

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{1}{3(x+h)} - \dfrac{1}{3x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x-(x+h)}{3x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-h}{3xh(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{3x(x+h)} \\ &= -\frac{1}{3x^2} \end{aligned}$$

(c) [5 points] Find the tangent line to the graph of $f(x) = \dfrac{1}{3x}$ at $x = 2$.

Solution By part (b), the slope of the tangent line at $x = 2$ is $f'(2) = -\frac{1}{3(2)^2} = -\frac{1}{12}$. Since $f(2) = \frac{1}{6}$, the line passes through the point $(2, \frac{1}{6})$. Its equation is therefore

$$y - \frac{1}{6} = -\frac{1}{12}(x - 2)$$

by the point-slope formula.

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6. [15 points] The world's population is about $P(t) = 6e^{0.013t}$ billion people, with t measured in years since 1999. Find $P'(17)$. Write a sentence or two explaining the meaning of your answer; be sure to include a discussion of units.

Solution By the chain rule

$$P'(t) = 6e^{0.013t} \cdot (0.013)$$

so $P'(17) = 6e^{0.013(17)} \cdot (0.013) \approx 0.0973$. This means that in 2016 the world population will be growing at a rate of about 97.3 million people per year.