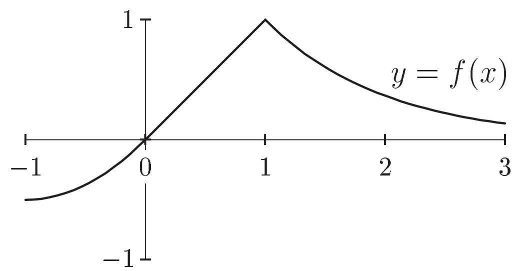
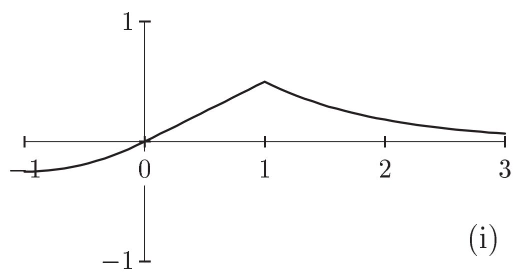
**College of the Holy Cross, Fall Semester, 2004**
**Math 131, Midterm 1 Solutions**

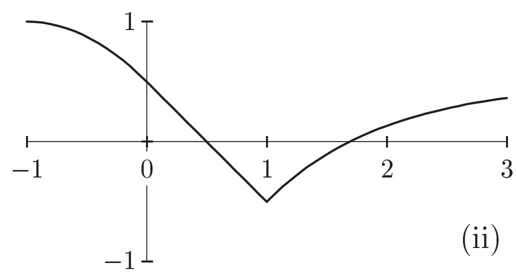
1. [5 points each] The graph $y = f(x)$ and four graphs obtained by transforming it are shown. Match the given formulas with the corresponding graph. Note that there is an extra graph.

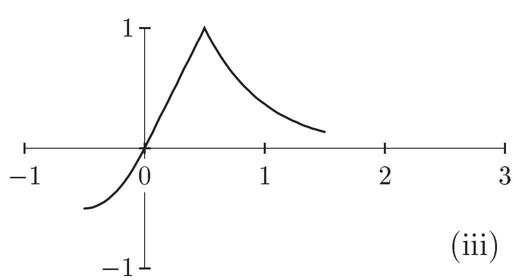


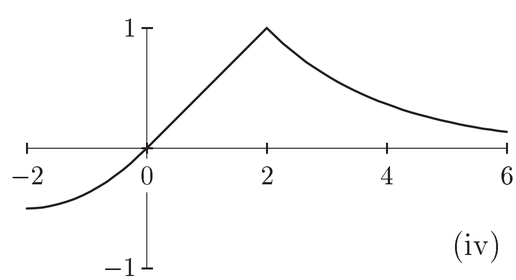

(a) $y = f(2x)$: (iii) (b) $y = \frac{1}{2}f(x)$: (i) (c) $y = \frac{1}{2} - f(x)$: (ii)

Graph (iv) has equation $y = f(\frac{1}{2}x)$.

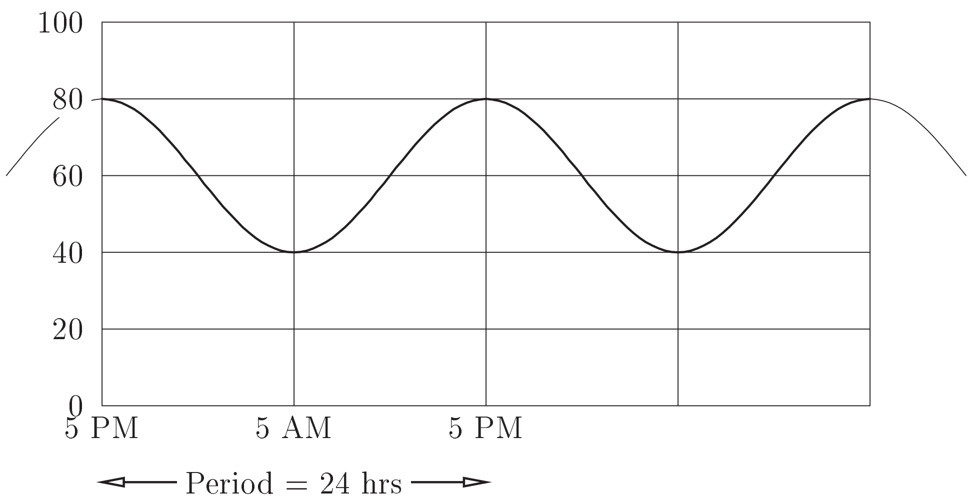


2. [20 points] The desert temperature $H$ varies sinusoidally from a high of 80°F at 5 PM to a low of 40°F at 5 AM. Find a formula for $H$ as a function of $t$, with $t$ measured in hours from 5 PM. You may use the graph below for reference; it's a good idea to start by labeling the vertical and horizontal scales.




**Solution** Because $t = 0$ at a "crest" (the temperature is maximum), we seek a function of the form $H(t) = D + A\cos(Bt)$ with $A > 0$ the amplitude of oscillation. The maximum minus minimum is $40 = 2A$, so $A = 20$. The vertical shift $D$ is the halfways point between the extremes, so $D = \frac{1}{2}(80 + 40) = 60$. Finally, the period is 24 hours, so $B = 2\pi/24 = \pi/12$. In summary

$$H(t) = 60 + 20\cos(\tfrac{\pi t}{12}).$$

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3. [10 points each] In each part, fill in the table as indicated.

(a) Assuming $f$ is a **linear function**.

**Solution** The line through the points $(-1, 24)$ and $(1, 12)$ has slope

$$m = (24 - 12)/(-1 - 1) = 12/-2 = -6.$$

The point-slope form of the equation is $(y - 12) = -6(x - 1)$, which simplifies to $y = 18 - 6x$. To fill in the table, set $x = 5$ and $x = 10$:

| $x$ | $-1$ | $1$ | $5$ | $10$ |
|---|---|---|---|---|
| $f(x)$ | $24$ | $12$ | $-12$ | $-42$ |

(b) Assuming $f$ is an **exponential function**.

**Solution** Here we are told to assume $f(x) = P_0 a^x$. Using the values provided gives two equations,

$$24 = P_0 a^{-1}, \qquad 12 = P_0 a.$$

Dividing the first equation by the second cancels $P_0$: $2 = 24/12 = a^{-1}/a = a^{-2}$, or $a = 2^{-1/2}$. Next, we plug this value into one of the equations (say the first) and solve for $P_0$: Since $24 = P_0 a^{-1}$, we have $P_0 = 24a = 24/\sqrt{2} = 12\sqrt{2}$. Thus

$$f(x) = P_0 a^x = (12\sqrt{2})2^{-x/2}.$$

As above, we fill in the table by setting $x = 5$ and $x = 10$.

| $x$ | $-1$ | $1$ | $5$ | $10$ |
|---|---|---|---|---|
| $f(x)$ | $24$ | $12$ | $3$ | $\frac{3\sqrt{2}}{8} \simeq 0.53$ |

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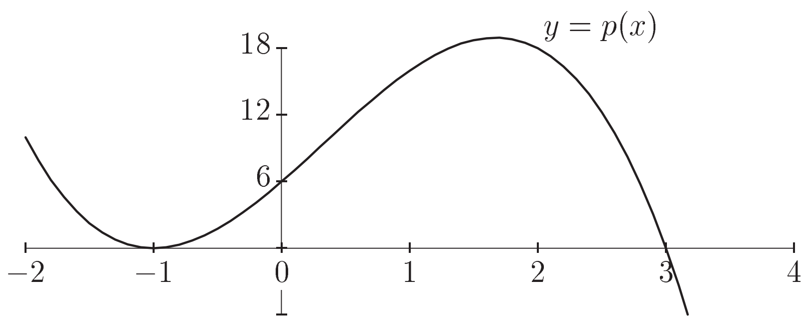
4. [10 points] Find the polynomial whose graph is shown; express your answer in **both** factored and expanded (multiplied out) form.



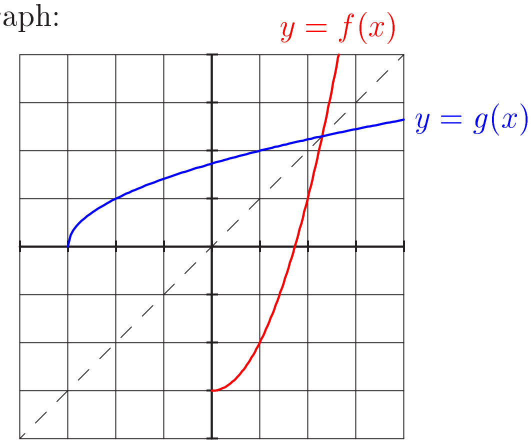

**Solution** We have $p(x) = k(x+1)^2(x-3)$. Since $p(0) = 6$, we find that $6 = k(1)^2(-3)$, or $k = -2$. Thus,

$$\begin{aligned} p(x) &= -2(x+1)^2(x-3) \qquad \text{(factored)} \\ &= -2(x^2+2x+1)(x-3) \\ &= -2(x^3 - x^2 - 5x - 3) \\ &= -2x^3 + 2x^2 + 10x + 6. \qquad \text{(expanded)} \end{aligned}$$

5. [10 points each] Let $y = f(x) = x^2 - 3$ for $x \geq 0$.

   (a) Find the inverse function in the form $y = f^{-1}(x)$, and find the domain of $f^{-1}$.

   **Solution** Formally, exchange $x$ and $y$, obtaining $x = y^2 - 3$, then solve for $y$. This gives $y = \pm\sqrt{x+3}$. To see whether we need a negative sign or not, it's easiest to consider the graph:




   We conclude that $y = f^{-1}(x) = \sqrt{x+3}$.

   (b) Suppose $g(x) = x + e^x$. With $f(x)$ as above, find $g\big(f(x)\big)$.

   **Solution** $g\big(f(x)\big) = g(x^2 - 3) = (x^2 - 3) + e^{(x^2-3)}$.

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6. [15 points] An automobile costs \$25,000 and depreciates in value by 20% per year. How many years pass before the car is worth \$5000? Give both an **exact answer** (using fractions, square roots, logarithms, etc. as needed) and a numerical answer rounded to two decimal places.

   **Solution** Let $V(t)$ be the value of the car in thousands of dollars after $t$ years. (Note the explicit choice of units. This should be your first step when creating a mathematical model!) Because $V(0) = 25$ and $V$ decreases to 0.8 times its current value after 1 year, we have

$$V(t) = 25(0.8)^t.$$

   The problem asks for the value $T$ so that $V(T) = 25(0.8)^T = 5$. Dividing by 25 and taking natural logarithms gives

$$(0.8)^T = 0.2, \qquad T\ln(0.8) = \ln(0.2), \qquad T = \frac{\ln(0.2)}{\ln(0.8)} \simeq 7.21 \text{ years}$$