Mathematics 131 – Calculus for Physical and Life Sciences 1
Lab Day 3 – Parametric Curves
November 1, 2004

Background

A parametrized curve in the plane gives the position of a particle or moving point as a function of time. For this we need to know both the x and y coordinates as functions of time:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

The form and shape of the curve "swept out" by the particle depends on the specific functions $x(t)$ and $y(t)$. In this lab you will use Maple to plot several parametrized curves that are not immediate to visualize, and/or difficult to draw by hand. In your lab report worksheet, hand in plots of all these parametric curves, and answer the questions in text regions.

Lab Work

A) Start by entering the command

```
plot([cos(t),sin(t),t=0..2*Pi],scaling=constrained);
```

to plot the parametric curve with coordinate functions $x(t) = \cos(t)$ and $y(t) = \sin(t)$. (Note the square brackets around the component functions and time interval. The "scaling" option forces Maple to use the same horizontal and vertical scales; *always include this today*.) What curve is this? Why? (Hint: What is true about $(x(t))^2 + (y(t))^2$ for all t?)

B) Plot the parametric curve with coordinate functions $x(t) = 2t/(t^2+1)$, $y(t) = (t^2-1)/(t^2+1)$ for $-100 \le t \le 100$. Identify this curve visually, and use algebra to show that your guess is correct.

C) Next, plot the parametric curve with component functions $x(t) = t - \sin t$, $y(t) = 1 - \cos t$ for $0 \le t \le 6\pi$. (As above π is `Pi` in Maple.) Find the values of t at which the velocity is zero. (Velocity is zero when $x'(t) = 0$ and $y'(t) = 0$.) Where on the curve are these points located?

D) The curve from question C is called a *cycloid*. It is the path traced by a point on the rim of a rolling wheel of radius 1. By changing the multiplier on the trig functions, you can simulate motion on other wheels. For example, $x(t) = t - 1.2\sin t$, $y(t) = 1 - 1.2\cos t$ might represent the path of a point on the rim of a train wheel, while $x(t) = t - 0.9\sin t$, $y(t) = 1 - 0.9\cos t$ might be the path of a bicycle reflector. (Plot these curves as well to see the changes.)

Comment: Unlike a function graph, a parametrized curve can even have cusps or corners at points where both coordinate functions are differentiable if the velocity is zero there. Intuitively, the point can come to a stop momentarily, then proceed in a different direction, thereby tracing a curve with no tangent line.

E) Plot $x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $0 \le t \le 2\pi$. Find all points on the curve where the velocity is zero. Where on the curve are these points located? Use algebra to show that this curve is given implicitly by $x^{2/3} + y^{2/3} = 1$.

F) Plot $x(t) = e^{t/2\pi} \cos t$, $y(t) = e^{t/2\pi} \sin t$, $-4 \le t \le 4$. Try graphing $x(t) = e^{kt} \cos t$, $y(t) = e^{kt} \sin t$ for a few values of k (both positive and negative). What happens as $k \to 0$? How do the curves with $k > 0$ differ from those with $k < 0$? Give examples of two natural forms (living organisms from at least two biological kingdoms) that this curve resembles.

Explanation: Curves in this family are known as *logarithmic spirals*, and have the property that they meet every radial line through the origin at a constant angle. This property explains their appearance in biological growth.

G) Plot $x(t) = t \cos t$, $y(t) = t \sin t$, $0 \le t \le 6\pi$.

Explanation: This curve is a *spiral of Archimedes*, and these can be used to trisect angles together with a compass and straightedge. It is a famous theorem that a general angle cannot be trisected using **only** a compass and straightedge.

H) Plot $x(t) = \cos(3t) \cos t$, $y(t) = \cos(3t) \sin t$, for $0 \le t \le 2\pi$. How many "petals" do you think $x(t) = \cos(5t) \cos t$, $y(t) = \cos(5t) \sin t$, $0 \le t \le 2\pi$, has? Plot the curve and see if your guess is correct.

What about $x(t) = \cos(2t) \cos t$, $y(t) = \cos(2t) \sin t$, for $0 \le t \le 2\pi$? Again, try it and see. If you can, explain the outcome of this experiment. How many petals do you think $x(t) = \cos(4t) \cos t$, $y(t) = \cos(4t) \sin t$ has?

Explanation: These curves are often called "rose curves", for the obvious reason that they look like stylized flowers.

I) Consider a circular ring of inner radius 96, inside of which is rolled a wheel of radius $r < 96$. A point on the edge of the wheel traces the "hypocycloid" parametrized by

$$\begin{cases} x(t) = (96 - r) \cos t + r \cos((96 - r)t/r) \\ y(t) = (96 - r) \sin t - r \sin((96 - r)t/r) \end{cases}$$

(The "Spirograph" toy you may have played with as a child draws these curves.) Plotting over $0 \le t \le 2\pi$ will usually not give the entire curve; use a large multiple like 0 to 100π for the plot interval. Plot the resulting curves for $r = 24$, 30, 35, 36, 40, 52, 56, 63, 64, 72, and 84.

Note: If you want to get a nice visual explanation for how these curves are generated, go to a web browser and look at the page `http://mathworld.wolfram.com/Hypocycloid.html` and scroll down the page until you see the animations.

Assignment

Lab writeups due in class on Monday, November 8.