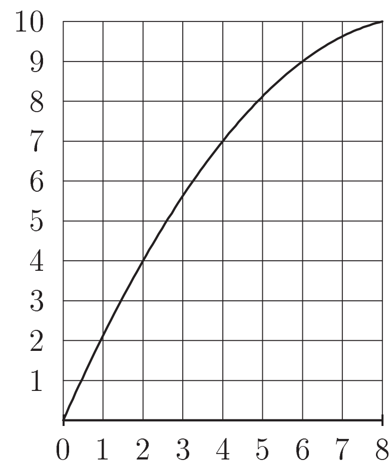
College of the Holy Cross, Spring Semester, 2005
Math 132, Midterm 2 Solutions
Thursday, March 31

1. The graph of the function $f(x)$ is shown below.



(a) [10 points] Use the graph of f to compute the following approximations of $\int_0^8 f(x)\,dx$.

Solution: Since $n = 2$, $\Delta x = \frac{8-0}{2} = 4$.

$LEFT(2) : \underline{\ 28\ } = 0 \cdot 4 + 7 \cdot 4$

$RIGHT(2) : \underline{\ 68\ } = 7 \cdot 4 + 10 \cdot 4$

$MID(2) : \underline{\ 52\ } = 4 \cdot 4 + 9 \cdot 4$

$TRAP(2) : \underline{\ 48\ } = \dfrac{1}{2}(LEFT(2) + RIGHT(2))$

$SIMP(2) : \underline{\ 50.6\ } = \dfrac{2MID(2) + TRAP(2)}{3}$

(b) [6 points] For each method, decide whether the approximation of $\int_0^8 f(x)\,dx$ is an *overestimate*, an *underestimate*, or that this *cannot be determined* from the given information. In the case of an overestimate or underestimate, briefly explain your reasoning.

Method	Type of Estimate	Reason
LEFT	underestimate	f is increasing
RIGHT	overestimate	f is increasing
MID	underestimate	f is concave down
TRAP	overestimate	f is concave down
SIMP	can't determine	

2. (a) [6 points] Set up an integral that represents the arclength of the portion of the graph of $y = \sin(x)$ between $x = 0$ and $x = \pi$. **Do not evaluate the integral.**

   Solution: Using the usual form of the arclength integral for the graph $y = f(x)$, we get $\frac{dy}{dx} = \cos(x)$ and

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx$$

   (b) [6 points] Approximate the integral in part (a) using a left hand sum with $n = 4$ subintervals.

   Solution: We have $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$. Then with the left hand sum approximation:

$$\begin{aligned}
\int_0^\pi \sqrt{1 + \cos^2 x}\, dx &\doteq \sqrt{1 + \cos^2(0)} \cdot \frac{\pi}{4} + \sqrt{1 + \cos^2\left(\frac{\pi}{4}\right)} \cdot \frac{\pi}{4} \\
&\quad + \sqrt{1 + \cos^2\left(\frac{\pi}{2}\right)} \cdot \frac{\pi}{4} + \sqrt{1 + \cos^2\left(\frac{3\pi}{4}\right)} \cdot \frac{\pi}{4} \\
&= \left(\sqrt{2} + \sqrt{3/2} + 1 + \sqrt{3/2}\right) \frac{\pi}{4} \\
&= \left(\sqrt{2} + \sqrt{6} + 1\right) \frac{\pi}{4} \\
&\doteq 3.82
\end{aligned}$$

3. Rewrite each of the following improper integrals as a limit, or limits. State whether the integral converges or diverges, and compute its value if it converges.

   (a) [6 points] $\displaystyle\int_{-1}^2 \frac{1}{x^3}\, dx$

   Solution: This integral is improper because the integrand is undefined at $x = 0$. Since this is in the interior of the interval of integration, we look at

$$\lim_{b \to 0^-} \int_{-1}^b x^{-3}\, dx + \lim_{a \to 0^+} \int_a^2 x^{-3}\, dx = \lim_{b \to 0^-} \frac{-1}{2} x^{-2} \Big|_0^b + \lim_{a \to 0^+} \frac{-1}{2} x^{-2} \Big|_a^2$$

   Both of these limits are undefined, so the improper integral *diverges*.

(b) [6 points] $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

Solution: This integral is improper because the integrand is not defined at the endpoint $x = 2$. We have

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx &= \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-x^2}}\,dx \\ &= \lim_{b\to 2^-} \arcsin\left(\frac{x}{2}\right)\Bigg|_0^b \qquad \text{\#28 in table} \\ &= \lim_{b\to 2^-} \arcsin\left(\frac{b}{2}\right) \\ &= \frac{\pi}{2}\end{aligned}$$

So the integral converges to the value $\frac{\pi}{2}$.

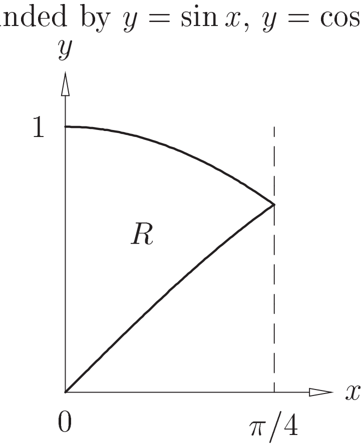
(c) [6 points] $\int_0^\infty e^{-4x}\,dx$

Solution: This integral is improper because of the infinite limit of integration.

$$\begin{aligned}\int_0^\infty e^{-4x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-4x}\,dx \\ &= \lim_{b\to\infty} \frac{-1}{4}e^{-4x}\Bigg|_0^b \\ &= \lim_{b\to\infty} \frac{1}{4}(1-e^{-4b}) \\ &= \frac{1}{4}\end{aligned}$$

So the integral converges to the value $\frac{1}{4}$.

4. Let R denote the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/4$.



(a) [7 points] Find the area of R.

Solution: Think of subdividing the area into vertical strips. The height of each is $\cos(x) - \sin(x)$ and the width is dx, so the area is given by

$$\int_0^{\pi/4} \cos(x) - \sin(x)\ dx = (\sin(x) + \cos(x))\Big|_0^{\pi/4} = \sqrt{2} - 1 \doteq .414$$

(b) [7 points] Find $\overline{x}$, the x-coordinate of the center of mass of R. (Assume the region has constant mass density $\delta = 1$.)

Solution: To find $\overline{x}$, we compute $Moment/Mass$. The mass was already computed in part a. The moment is given by the integral

$$\int_0^{\pi/4} x(\cos(x) - \sin(x))\ dx$$

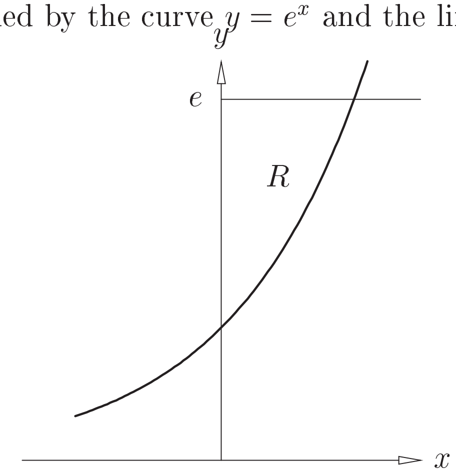
To integrate here we use parts with $u = x$, and $dv = \cos(x) - \sin(x)$. Then $du = dx$, and $v = \sin(x) + \cos(x)$. This gives

$$\begin{aligned}\int_0^{\pi/4} x(\cos(x) - \sin(x))\ dx &= x(\sin(x) + \cos(x))\Big|_0^{\pi/4} - \int_0^{\pi/4} \sin(x) + \cos(x)\ dx \\ &= x(\sin(x) + \cos(x))\Big|_0^{\pi/4} + (\cos(x) - \sin(x))\Big|_0^{\pi/4} \\ &= \frac{\pi\sqrt{2}}{4} - 1\end{aligned}$$

Then the center of mass is at

$$\overline{x} = \frac{\frac{\pi\sqrt{2}}{4} - 1}{\sqrt{2} - 1} \doteq .267$$

5. Let R be the region bounded by the curve $y = e^x$ and the lines $x = 0$ and $y = e$.



(a) [8 points] Find the volume of the solid obtained by revolving R about the x-axis.

Solution: Rotating around the x-axis, we get a solid with "washer" cross sections because of the gap between the region R and the x-axis. The inner radius for the cross section at x is $r_{in} = e^x$ while the outer radius is $r_{out} = e$. The upper limit of integration is found by setting $e^x = e$ and solving for x: $x = 1$. The volume is

$$\int_0^1 \pi e^2 - \pi e^{2x}\, dx = \pi \left(e^2 x - \frac{e^{2x}}{2} \right) \Bigg|_0^1 = \frac{\pi}{2}(e^2 + 1) \doteq 13.18$$

(b) [6 points] Set up an integral that represents the volume of the solid whose base is R and whose cross-sections perpendicular to the x-axis are squares with base lying in R. **Do not evaluate the integral.**

Solution: By Cavalieri's Principle, we integrate the cross sectional area to get the volume. Since the cross section at x is a square with side $e - e^x$, the cross sectional area is $(e - e^x)^2$, and the volume is

$$\int_0^1 (e - e^x)^2\, dx$$

6. Suppose a metal rod of length 2 meters has a mass density $\delta(x) = 5 + 1.2x^2$ kg per meter.

(a) [7 points] Find the total mass of the rod.

Solution: Since the rod is a straight thin (essentially 1-dimensional) object, the mass of a small piece is approximately density value times the length. Adding up these terms, we get:

$$M = \int_0^2 5 + 1.2x^2\, dx = (5x + .4x^3) \Big|_0^2 = 13.2\text{kg}$$

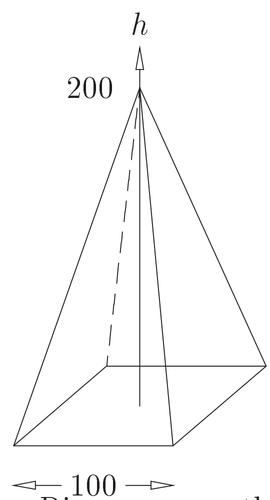
(b) [7 points] Find the center of mass of the rod.

Solution: The center of mass is at

$$\begin{aligned}
\overline{x} &= \frac{1}{13.2}\int_0^2 x(5+1.2x^2)\ dx \\
&= \frac{1}{13.2}\int_0^2 5x + 1.2x^3\ dx \\
&= \frac{1}{13.2}\left(\frac{5x^2}{2} + .3x^4\Big|_0^2\right) \\
&= \frac{14.8}{13.2} \\
&\doteq 1.12
\end{aligned}$$

(Note: This is to the right of the midpoint of the rod, which is what we expect since the density is increasing with x. There is more mass to the right of the midpoint than to the left.

7. A 200 meter tall pyramid has a square base with side length 100 meters. Its mass density (in kg/m^3) at height h meters above the base is given by $\delta(h) = 1 + 0.01h$.



(a) [6 points] Write down a Riemann sum that approximates the total mass of the pyramid.

Solution: Slicing the pyramid horizontally, the slice at height h above the base is approximately a rectangular solid with length and width $s(h)$ depending on h, and height Δh. The side $s(h)$ is found by similar triangles:

$$s(h) = \frac{1}{2}(200 - h).$$

So we add up (density) $\cdot$ (volume) for all the slices:

$$M \doteq \sum (1 + .01h) \left(\frac{1}{2}(200 - h) \right)^2 \Delta h$$

(b) [6 points] Using the answer to part (a), write down an integral that equals the mass of the pyramid. **Do not evaluate the integral.**

Solution:

$$M = \int_0^{200} (1 + .01h) \cdot \frac{1}{4}(200 - h)^2 \, dh$$