Mathematics 131 – Calculus for Physical and Life Sciences
Discussion 3 – Optimization
November 9, 2004

Background

Yesterday in class we discussed critical points of a function f ($x = p$ in the domain of f where either $f'(p) = 0$ or $f'(p)$ does not exist). We also introduced the First Derivative Test for deciding whether a critical point is a local maximum, a local minimum, or neither.

The information from the *second derivative* can also be used to decide this question.

Second Derivative Test: If $f'(p) = 0$ and f' is continuous at p, then

- $f''(p) > 0$ implies the graph $y = f(x)$ must be concave up at p, so p is a *local minimum*,
- $f''(p) < 0$ implies the graph $y = f(x)$ must be concave down at p, so p is a *local maximum*,
- $f''(p) = 0$ implies the test is inconclusive (f could have a local maximum, a local minimum, or neither at p.

"Optimization" refers to the problem of finding the maximum and/or minimum value of a function on some interval. For *"closed"* intervals $[a, b]$, it is a theorem from analysis that continuous functions attain their maximum and minimum values at some point in the interval, possibly at one of the endpoints. To find the maximum and minimum values, we can find all critical points, test them using the First or Second Derivative Tests, then compare the function values at the critical points with the values $f(a)$ and $f(b)$, selecting the largest and smallest f-values. You will do this in questions B, C below.

Note: No graphing calculators on this one!

Discussion Questions

A) (The "inconclusive" case of the Second Derivative Test.) Give three examples of functions $f(x)$ with critical points at $x = 0$ for which $f'(0) = 0$ and $f''(0) = 0$. In your first example, make f have a local maximum at $x = 0$, in your second example, make f have a local minimum at $x = 0$. In your final example, make f have neither a maximum nor a minimum at $x = 0$.

B) Consider the function

$$f(x) = \frac{x^2}{x^2 - 1}$$

1) Compute $f'(x)$ and determine the sign diagram for $f'(x)$. (Careful: The graph $y = f(x)$ has vertical asymptotes; that means $f'(x)$ will also be undefined at some points. You will need to take those into account too.)
2) Compute $f''(x)$ and determine the sign diagram for $f''(x)$. If we require inflection points to be points where f and f' are continuous, does this function have any inflection points? Why or why not?

3) Using the information from parts 1 and 2, sketch the graph $y = f(x)$.
4) Suppose we wanted to determine the maximum and minimum values of $f(x)$ on the interval $[-1/2, 1/3]$. Here is one process we could follow:
   a) First find all critical points in the interval. What are they here?
   b) Compute $f(p)$ for any critical points p and compare with $f(-1/2)$ and $f(1/3)$.
   c) The largest number you find will be the maximum on the interval. What is it here?
   d) The smallest number you find will be the minimum on the interval. What is it here? Is the minimum on $[-1/2, 1/3]$ a *local minimum* for f? Why or why not?
5) Repeat part 4 with the same $f(x)$ but the interval $[3/2, 4]$. Find the maximum and minimum values of f on that interval.

C) An electric current I in amperes is given as a function of t by

$$I(t) = \cos(wt) + \sqrt{3}\sin(wt)$$

where w is a constant (the frequency).

1) With $w = \pi/2$, find the maximum and minimum values of I for t in the interval $[0, 2]$. Follow the process outlined in B 4 above.
2) Letting w be arbitrary now, what are the maximum and minimum values *overall* of I (that is taking all t into account)? Do your maximum and minimum values depend on w? Do the ts where the maximum and minimum I values are attained depend on w? (These are sometimes called the *global* maximum and minimum values.)

Assignment

Group writeups due in class on Monday, November 15.