**College of the Holy Cross, Spring Semester, 2005**
**Math 132, Midterm 3 Solutions**
**April 27, 2005**

1. [5 points each] Suppose $f(0) = 0.5$, $f'(0) = -1$ and $f''(0) = -2$.

   (a) Write down the Taylor polynomial of degree 2 for $f$ near $a = 0$.

   **Solution:** The polynomial is

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 0.5 - x - x^2$$

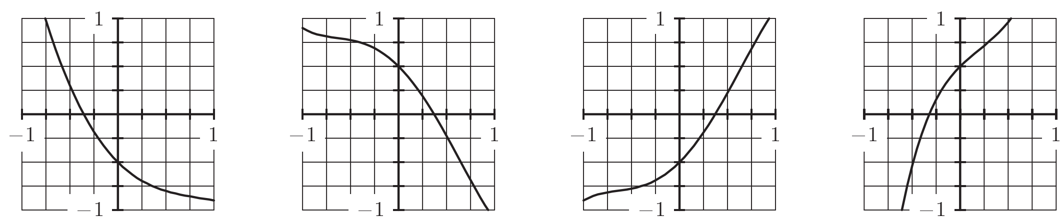
   (b) Use your answer to part (a) to estimate $f(0.3)$.

   **Solution:** Substitute $x = 0.3$ in $p_2(x)$:

$$p_2(0.3) = 0.5 - 0.3 - (0.3)^2 = .11$$

   (c) Which (if any) of the following could be the graph of $f$? (Recall that $f(0) = 0.5$, $f'(0) = -1$ and $f''(0) = -2$. More than one answer may be correct.)

   ◯ (ii)




(i) (ii) (iii) (iv)

   **Solution:** At $a = 0$, the function should be positive ($f(0) > 0$), decreasing ($f'(0) < 0$), and concave down ($f''(0) < 0$). Only (ii) is all three.

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2. [10 points] Evaluate (find the sum of) the series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \frac{80}{81} - \cdots$

   **Solution:** This is a geometric series with $a = 5$ and $r = \frac{-2}{3}$. Since $-1 < r < 1$, the series is convergent and the sum is

$$\frac{a}{1-r} = \frac{5}{1+\frac{2}{3}} = 3.$$

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3. [10 points] Use the power series for $e^x$ about $a = 0$ to find the power series for $ze^{-z^2}$ about $a = 0$. Express your answer both in summation form, and by writing out the first four nonzero terms.

**Solution:** The Taylor series for $e^x$ about $a = 0$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

To get the series for $ze^{-z^2}$, we substitute $x = -z^2$, and multiply the result by $z$ to obtain:

$$z\sum_{k=0}^{\infty} \frac{(-z^2)^k}{k!} = z\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{k!} = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{k!}$$

The first four nonzero terms are

$$z - z^3 + \frac{z^5}{2!} - \frac{z^7}{3!}.$$

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4. The distribution of length in a certain population of inchworms is described by the pdf $p(x) = c(3x^2 - x^3)$ for $0 \le x \le 3$.

(a) [8 points] Find the value of the constant $c$.

**Solution:** The total area under the graph of a density must be 1, so

$$\begin{aligned} 1 &= c\int_0^3 3x^2 - x^3 \, dx \\ &= c\left(x^3 - \frac{x^4}{4}\right|_0^3 \\ &= c\frac{27}{4} \\ \Rightarrow c &= \frac{4}{27} \end{aligned}$$

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(b) [7 points] Find the proportion of inchworms with length at most 2 cm.

**Solution:** This proportion is computed by the integral $\int_0^2 p(x)\, dx$,

which is $\frac{4}{27}\int_0^2 3x^2 - x^3 \, dx = \frac{16}{27} \doteq .593$

(about 59.3% of the inchworms).

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5. [5 points each] Use the indicated test to determine whether the given series converges or diverges. **Justify your answers.**

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+10}$; integral test

**Solution:** We apply the substitution $u = x^3 + 10$ to the integral

$$\int \frac{x^2\,dx}{x^3+10} = \int \frac{\frac{1}{3}\,du}{u} = \frac{1}{3}\ln(u) = \frac{1}{3}\ln(x^3+10) + C.$$

Then $\int_1^b \frac{x^2\,dx}{x^3+10} = \ln(b^3+10) - \ln(11)$. Taking $\lim_{b\to\infty} \ln(b^3+10) - \ln(11)$, we do not get a finite value so the integral diverges, and so does the series.

○ Diverges

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(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n^3+3n^2+5}$; comparison test

**Solution:** For all $n \geq 1$:

$$\frac{n}{4n^3+3n^2+5} < \frac{n}{4n^3} = \frac{1}{4n^2} < \frac{1}{n^2}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the $p$-series with $p = 2 > 1$. Hence it is convergent. By the comparison test, the given series converges too.

○ Converges

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6. [15 points] Let $f(x) = \sqrt{x}$. Use **the definition** to calculate its Taylor polynomial of degree 3 at $a = 1$.

**Solution:** We have $f(1) = 1$. $f'(x) = \frac{1}{2}x^{-1/2}$, so $f'(1) = \frac{1}{2}$. $f''(x) = \frac{-1}{4}x^{-3/2}$, so $f''(1) = \frac{-1}{4}$. Finally, $f'''(x) = \frac{3}{8}x^{-5/2}$, so $f'''(1) = \frac{3}{8}$. The Taylor polynomial is

$$f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{6}(x-1)^3 = 1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3.$$

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7. Both parts refer to the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n \cdot 5^n}$.

(a) [10 points] Use the ratio test to find the radius of convergence.

**Solution:**

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)\cdot 5^{n+1}}\cdot\frac{n\cdot 5^n}{(x-1)^n}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{5}\cdot|x-1| = \frac{1}{5}|x-1|.$$

So the radius of convergence is $R = \frac{1}{\frac{1}{5}} = 5$.

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(b) [5 points] Investigate the endpoint behavior, and determine the interval of convergence.

**Solution:** At $x = 1 + R = 6$, $\displaystyle\sum_{n=1}^{\infty}\frac{(6-1)^n}{n\cdot 5^n} = \sum_{n=1}^{\infty}\frac{1}{n}$. This is the harmonic series ($p$ series with $p = 1$). So it diverges. At $x = 1 - R = -4$, we get $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, which converges by the alternating series test. The interval of convergence is $-4 \le x < 6$.

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8. [5 points each] Suppose

$$p(x) = \frac{4}{\pi}\cdot\frac{1}{1+x^2}$$

is the probability density function for the quantity $x$, for $0 \le x \le 1$.

(a) Find the mean of $x$.

**Solution:** The mean is given by

$$\int_{-\infty}^{\infty} xp(x)\ dx = \frac{4}{\pi}\int_0^1 \frac{x}{1+x^2}\ dx = \frac{4}{\pi}\cdot\frac{1}{2}\ln(1+x^2)|_0^1 = \frac{2\ln(2)}{\pi} \doteq .4413$$

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(b) Find the median of $x$.

**Solution:** The median is the $t$ such that

$$\frac{1}{2} = \frac{4}{\pi}\int_0^t \frac{dx}{1+x^2} = \arctan(t)$$

This says

$$\arctan(t) = \frac{\pi}{8}$$

or $t = \tan\left(\frac{\pi}{8}\right) = \doteq .4142$.