College of the Holy Cross, Fall Semester, 2004
Math 132, Midterm 1 Solutions (All Sections)

1. [5 points each] Compute the following:

(a) $\displaystyle\int 2x(x^2+5)^{7/2}\,dx$

Solution Substitute for the function inside the power: $u = x^2+5$. Then $du = 2x\,dx$ and the integral becomes $\int u^{7/2}\,du$. By the Power Rule, this equals

$$\frac{2}{9}u^{9/2} + C = \boxed{\frac{2}{9}(x^2+5)^{9/2} + C}$$

(b) $\displaystyle\int \sin(3\theta)\,d\theta$

Solution Substitute for the function inside the sine: let $u = 3\theta$. Then $du = 3\,d\theta$ and the integral is

$$\frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u) + C = \boxed{-\frac{1}{3}\cos(3\theta) + C}$$

(c) $\displaystyle\int \frac{dx}{2x+1}$

Solution In this integral, if we let u be the denominator $u = 2x+1$, then $du = 2\,dx$ and the integral becomes

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \boxed{\tfrac{1}{2}\ln|2x+1| + C}$$

(d) $F'(x)$, where $F(x) = \displaystyle\int_\pi^x \frac{t}{1+\sin^2 t}\,dt$

Solution For this problem, we use the second Fundamental Theorem of Calculus, which says that if $f(t)$ is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Applying this here, we get $\boxed{F'(x) = \dfrac{x}{1+\sin^2 x}}$

2. [10 points] The graph $y = f(x)$ is shown. Let F be the antiderivative of f satisfying $F(2) = 0$. Find the indicated values, and carefully sketch the graph $y = F(x)$ in the grid provided.

$F(1) =$ ______ $F(3) =$ ______ $F(4) =$ ______ $F(6) =$ ______

Solution By the first part of the Fundamental Theorem of Calculus, and the information given in the graph of $y = f(x)$ about areas between the graph and the x-axis,

$$5 = \int_1^2 f(x)\,dx = F(2) - F(1) = 0 - F(1) \Rightarrow \boxed{F(1) = -5}$$

$$6.25 = \int_2^3 f(x)\,dx = F(3) - F(2) = F(3) - 0 \Rightarrow \boxed{F(3) = 6.25}$$

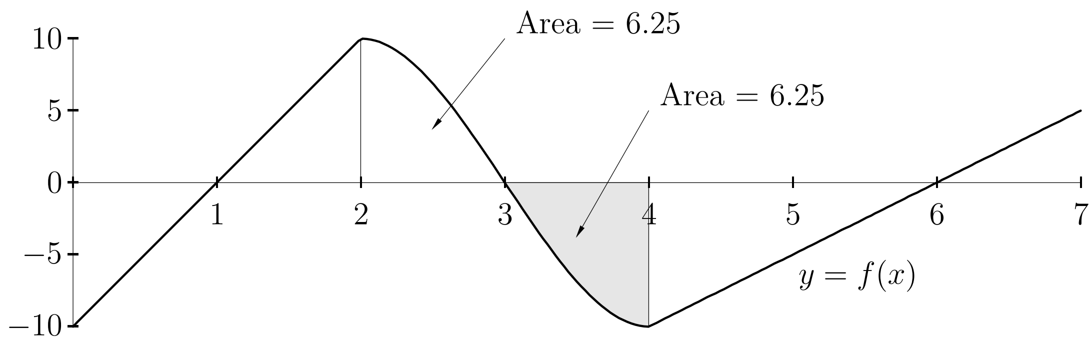
$$-6.25 = \int_3^4 f(x)\,dx = F(4) - F(3) = F(4) - 6.25 \Rightarrow \boxed{F(4) = 0}$$

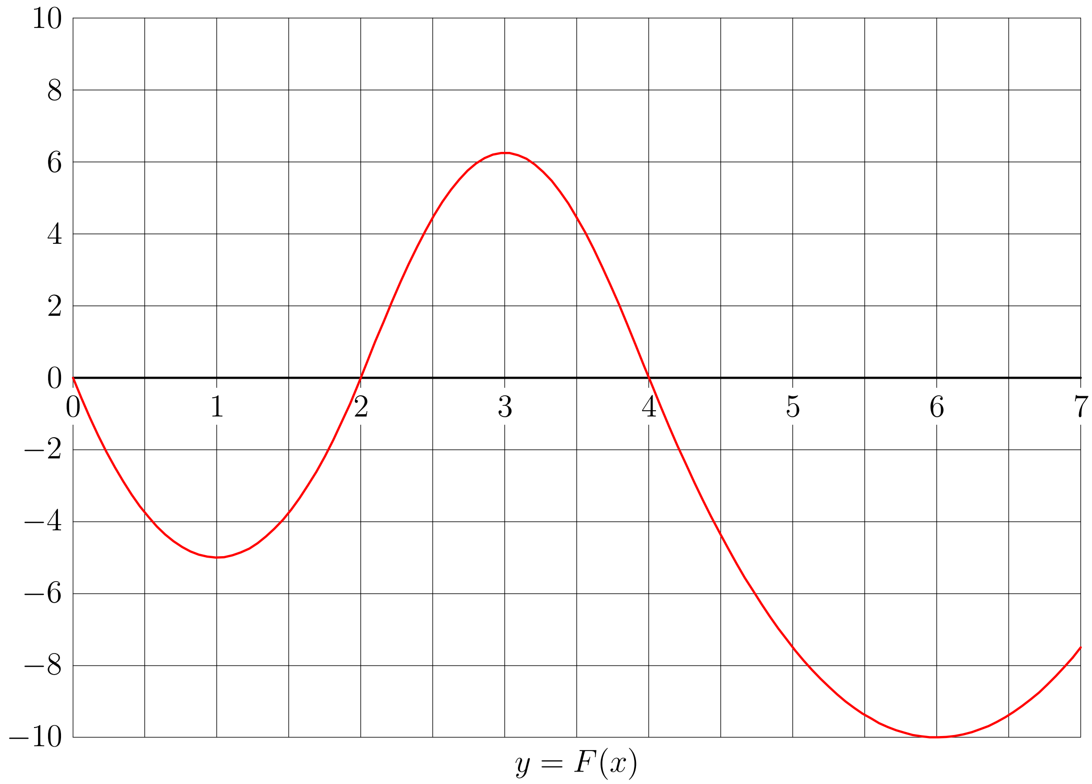
$$-10 = \int_4^6 f(x)\,dx = F(6) - F(4) = F(6) - 0 \Rightarrow \boxed{F(6) = -10}$$

Similarly, we get

$$-5 = \int_0^1 f(x)\,dx = F(1) - F(0) = -5 - F(0) \Rightarrow F(0) = 0$$

$F(x)$ is increasing on intervals where $f(x)$ is positive and decreasing on intervals where $f(x)$ is negative. It is concave up on intervals where $f(x)$ is increasing and concave down on intervals where $f(x)$ is decreasing. $F(x)$ has critical points when $f(x) = 0$. The graph $y = F(x)$ is shown on the next page.


Area = 6.25
Area = 6.25
10
5
0
−5
−10
1
2
3
4
5
6
7
y = f(x)


10
8
6
4
2
0
−2
−4
−6
−8
−10
0
1
2
3
4
5
6
7
y = F(x)

3. [5 points each] A drag racer accelerates at $a(t) = (20 + t)$ ft/sec^2.

(a) How fast is the car traveling after 6 seconds? (The car is initially at rest.)

Solution The acceleration is not constant, so the book formulas $v(t) = v_0 + at$ and $x(t) = v_0 t + \frac{1}{2}at^2$ do not apply. The velocity is the integral of the acceleration:

$$v(t) = \int (20 + t)\, dt = 20t + \frac{t^2}{2} + C.$$

Since the car starts from rest, $v(0) = 0$, or $C = 0$. The velocity at time $t = 6$ is $v(6) = 120 + \frac{36}{2} = \boxed{138 \text{ ft/sec}}$

(b) How far has the car traveled after 6 seconds?

Solution For this we need the position $s(t)$ of the car as a function of time. The position will be the integral of the velocity, so

$$s(t) = \int \left[20t + \frac{t^2}{2}\right] dt = 10t^2 + \frac{t^3}{6} + C$$

for some constant. Now we may as well measure distances from the starting position of the car, so we can take $s(0) = 0$ also. This means this constant of integration is also zero: $C = 0$. Then the distance traveled by the car after 6 seconds is

$$s(6) - s(0) = 360 + 36 = \boxed{396 \text{ ft}}$$

Note: we could also set up this part as computing the *definite* integral $\displaystyle\int_0^6 v(t)\, dt$.

---

4. [10 points each] Find the integrals. Use the indicated method to get started.

(a) $\displaystyle\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx$ (Substitution)

Solution We substitute $u = 1 + \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}\, dx$. Converting the limits of integration as well ($x = 1$ gives $u = 2$ and $x = 4$ gives $u = 1 + \sqrt{4} = 3$), we get:

$$\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx = 2\int_2^3 \sqrt{u}\, du = \frac{4}{3}u^{3/2}\Big|_2^3 = \boxed{\frac{4}{3}\left(3\sqrt{3} - 2\sqrt{2}\right) \simeq 3.157\ldots}$$

(b) $\int t \sec^2 t\, dt$ (Integration by parts)

Solution The good choice here is to let $u = t$ and $dv = \sec^2 t$. Then $du = dt$ and $v = \tan t$. The parts formula gives

$$\int t \sec^2 t\, dt = t \tan t - \int \tan t\, dt = \boxed{t \tan t + \ln|\cos t| + C}$$

(For the last step, either use Formula 7 in the table or notice that $\tan t = \frac{\sin t}{\cos t}$ so the remaining integral has the form $-\int \frac{dw}{w}$ for $w = \cos t$.)

---

5. [10 points each] Compute the integrals

(a) $\int \frac{dx}{(4 - x^2)^{3/2}}$

Solution The $(4 - x^2)^{3/2}$ in the integrand indicates that we want the trigonometric substitution $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$, and the integral becomes

$$\int \frac{dx}{(4 - x^2)^{3/2}} = \int \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{4}\tan\theta + C = \boxed{\frac{1}{4}\frac{x}{\sqrt{4 - x^2}} + C}$$

The integral may be done either with Differentiation Formula 12 or Integral Formula 21 with $n = 2$. For the last step, converting back to functions of x, use the triangle with side opposite θ equal to x and hypotenuse 2. The adjacent side is $\sqrt{4 - x^2}$.

(b) $\int \frac{dx}{x^2 + 4x + 8}$

Solution Complete the square in the denominator: $x^2 + 4x + 8 = (x + 2)^2 + 4$. Then

$$\int \frac{dx}{x^2 + 4x + 8} = \int \frac{dx}{(x + 2)^2 + 4} = \int \frac{du}{u^2 + 4} \qquad \text{Letting } u = x + 2$$

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C \qquad \text{Formula 24}$$

$$= \boxed{\frac{1}{2}\arctan\left(\frac{x + 2}{2}\right) + C}$$

Note: This could also be done by the substitution $x + 2 = \tan\theta$ after the first step of completing the square.

6. [10 points] Find the indefinite integral $\displaystyle\int \frac{x^2+1}{x(x+2)(x-1)}\,dx$

Solution This is a rational function which does not simplify or yield to any obvious substitutions. So we apply the partial fraction method. From the factorization of the denominator,

$$\frac{x^2+1}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

for some A, B, and C. To determine those coefficients, clear denominators:

$$x^2+1 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$$

For one like this, the easiest way to determine values of A, B, C is to substitute x-values that zero out terms. With $x=0$, we get $1=-2A$, so $A=-\frac{1}{2}$. With $x=1$, $2=3C$, so $C=\frac{2}{3}$. With $x=-2$, $5=6B$, so $B=\frac{5}{6}$. Thus

$$\begin{aligned}\int \frac{x^2+1}{x(x+2)(x-1)}\,dx &= -\frac{1}{2}\int\frac{dx}{x} + \frac{5}{6}\int\frac{dx}{x+2} + \frac{2}{3}\int\frac{dx}{x-1} \\ &= \boxed{-\frac{1}{2}\ln|x| + \frac{5}{6}\ln|x+2| + \frac{2}{3}\ln|x-1| + C}\end{aligned}$$

---

7. [10 points] Compute any **two** of the following; clearly mark your choices and indicate the method of integration.

(i) $\displaystyle\int \frac{1}{\sqrt{1-x^4}}\,dx$ (ii) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ (iii) $\displaystyle\int \frac{x^2}{\sqrt{1-x^4}}\,dx$ (iv) $\displaystyle\int \frac{x^3}{\sqrt{1-x^4}}\,dx$

Solution The two "easy" ones here are (iv) and (ii). In (iv), the substitution $u = 1-x^4$ makes $du = -4x^3\,dx$ so the integral is

$$-\frac{1}{4}\int u^{-1/2}\,du = -\frac{1}{2}u^{1/2} + C = \boxed{-\frac{1}{2}\sqrt{1-x^4} + C}$$

In (ii), if you let $u = x^2$, then $du = 2x\,dx$ and the form is

$$\frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u = \boxed{\frac{1}{2}\arcsin(x^2) + C}$$