1. [5 points each] Circle the number of the graph showing each of the following functions.

- (a) \( f(x) = 3 - e^{-x} \) \hspace{1cm} \text{Answer: V}
- (b) \( f(x) = x^3 - 2x^2 - x + 2 \) \hspace{1cm} \text{Answer: IV}
- (c) \( f(x) = 4 \sin(\pi x) \) \hspace{1cm} \text{Answer: I}
- (d) \( f(x) = 4 - 4 \cos(\pi x) \) \hspace{1cm} \text{Answer: III}

- Note: Graph II is \( y = e^{-x} \). V is what you get from II, if that graph is reflected across \( x \)-axis and shifted up by 3.
2. [20 points] One of the functions given in the following table is linear and the other is exponential. Find formulas of the appropriate type for each.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>1.2</td>
<td>0.6</td>
<td>0.3</td>
<td>0.15</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>−2.3</td>
<td>−0.6</td>
<td>1.1</td>
<td>2.8</td>
</tr>
</tbody>
</table>

**Solution:** $g(x)$ is linear, since the slopes of the lines between all pairs of points in the table is 1.7. For instance using the first two points: $m = \frac{-0.6+2.3}{2-1} = 1.7$. The equation for $g$ is obtained by the point-slope form: $y - (-2.3) = 1.7(x - 1)$, so $y = 1.7x - 4$.

Since the problem said one is exponential, that means that $f(x)$ is the exponential one. We can find the equation for $f(x) = ca^x$ as usual. From the table with $x = 1, 2$, $ca^1 = 1.2$ and $ca^2 = 0.6$ so $\frac{ca^2}{a} = a = \frac{0.6}{1.2} = \frac{1}{2}$. Then from the first data point $c\frac{1}{2} = 1.2$, so $c = 2.4$. $g(x) = \frac{2.4}{2^x}$.

3.

- (a) [15 points] The depth of water in a tank oscillates sinusoidally once every 4 hours. The smallest depth is 2 feet and the maximum depth is 5 feet, which occurs at $t = 0$. Find a formula for the depth $d(t)$ if $t$ is the time in hours.

  **Solution:** The amplitude is $\frac{5-2}{2} = \frac{3}{2}$. The vertical shift is $\frac{5+2}{2} = \frac{7}{2}$. Putting $t = 0$ at the start of the period where there is a maximum means we want to use cos. Finally the period is 4, so we get

  $$d(t) = \frac{3}{2} \cos \left( \frac{\pi t}{2} \right) + \frac{7}{2}$$

- (b) [5 points] How fast is the depth changing at $t = 1.3$ hours? Is it increasing or decreasing?

  **Solution:** The question is asking for the derivative of $d(t)$ at $t = 1.3$: $d'(t) = -\frac{3\pi}{4} \sin \left( \frac{\pi t}{2} \right)$, so $d'(1.3) = -\frac{3\pi}{4} \sin \left( \frac{13\pi}{2} \right) \approx -2.1$ feet per hour. Since this is negative, the depth is decreasing at $t = 1.3$. 

4. Compute the following limits [5 points each]. Any legal method is OK.

- (a) **Solution:**

\[
\lim_{x \to 1} \frac{x^2 + 1}{x - 3} = \lim_{x \to 1} \frac{x^2 + 1}{x - 3} \\
= \frac{2}{-2} \\
= -1
\]

- (b) **Solution:** This limit is indeterminate of the form 0/0, so we use L’Hopital’s Rule – twice!

\[
\lim_{x \to 2} \frac{x^2 - 4x + 4}{\cos(x - 2) - 1} = \lim_{x \to 2} \frac{2x - 4}{-\sin(x - 2)} \\
= \lim_{x \to 2} \frac{2}{-\cos(x - 2)} \\
= -2
\]

- (c) This one can also be done by L’Hopital, or by some algebraic “trickery”:

\[
\lim_{x \to \infty} \frac{5x^2 - x + 21}{3x^2 + x + 1} = \lim_{x \to \infty} \frac{5x^2 - x + 21}{3x^2 + x + 1} \cdot \frac{1/x^2}{1/x^2} \\
= \lim_{x \to \infty} \frac{5 - 1/x + 21/x^2}{3 + 1/x + 1/x^2} \\
= \frac{5}{3}
\]
5.

- (a) [5 points] State the limit definition of the derivative. *Solution:*

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

(provided that this limit exists).

- (b) [10 points] Use the definition to compute \( f'(x) \) for \( f(x) = \sqrt{x} \).

*Solution:*

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}
= \lim_{h \to 0} \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})}
= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}}
= \frac{1}{2\sqrt{x}}
\]

- (c) [10 points] Find the equation of the tangent line to the graph \( y = \sqrt{x} \) at the point \([4, 2]\).

*Solution:* From part B, the derivative of \( \sqrt{x} \) at \( x = 4 \) is \( \frac{1}{2\sqrt{4}} = \frac{1}{4} \). This is the slope of the tangent line, which has the equation \( y - 2 = \frac{1}{4}(x - 4) \), or \( y = \frac{1}{4}x + 1 \).
6. Compute the following derivatives using the derivative rules. You need not simplify. [5 points each]

- (a) \( f(t) = t^3 - \frac{1}{\sqrt{t}} + \pi t \).

  \textit{Solution:} By the power and exponential rules,
  \[
  f'(t) = 3t^2 + \frac{1}{4}x^{-5/4} + \pi t \ln(\pi)
  \]

- (b) \( g(x) = \frac{x^2 - 2}{\cos(x) + 1} \)

  \textit{Solution:} By the quotient rule,
  \[
  g'(x) = \frac{(\cos(x) + 1)(2x) - (x^2 - 2)(-\sin(x))}{(\cos(x) + 1)^2}
  \]

- (c) \( h(z) = \ln(4z^2 + 2e^{\arctan(z)}) \)

  \textit{Solution:} By the chain rule,
  \[
  h'(z) = \frac{8z + \frac{2\arctan(z)}{1+z^2}}{4z^2 + 2e^{\arctan(z)}}
  \]

- (d) Find \( \frac{dy}{dx} \) if \( x^2y - 2y^3 = 3 \).

  \textit{Solution:} Differentiating implicitly, \( x^2 \frac{dy}{dx} + 2xy - 6y^2 \frac{dy}{dx} = 0 \). Solving for \( \frac{dy}{dx} \), we have
  \[
  \frac{dy}{dx} = \frac{-2xy}{-6y^2 + x^2}
  \]
7. Consider the family of curves defined by \( y = f(x) = x^4 + 2ax^2 \), where \( a \) is any fixed real number.

- (a) [10 points] Find the critical points of \( f \), construct a sign diagram for \( f'(x) \) in the case that \( a < 0 \). Which of your critical points are local maxima and which are local minima?

\[ \text{Solution:} \quad f'(x) = 4x^3 + 4ax = 4x(x^2 + a). \] When \( a < 0 \), this has three solutions, so three critical points: \( x = -\sqrt{-a}, 0, \sqrt{-a} \). (Note: \( a < 0 \) means \(-a > 0\), so \( \sqrt{-a} \) does exist in the real numbers in this case.) The sign diagram should show \( f' < 0 \) for \( x < -\sqrt{-a}, f' > 0 \) for \(-\sqrt{-a} < x < 0 \), \( f' < 0 \) for \( 0 < x < \sqrt{-a} \) and \( f' > 0 \) for \( x > \sqrt{-a} \). This means that \( x = \pm \sqrt{-a} \) are local minima and \( x = 0 \) is a local maximum (First Derivative Test).

- (b) [10 points] Repeat part a, but assume now that \( a > 0 \).

\[ \text{Solution:} \quad \text{As before, } f'(x) = 4x^3 + 4ax = 4x(x^2 + a). \] But when \( a > 0 \), the equation \( x^2 + a = 0 \) has no real solutions. So there is only one critical point: \( x = 0 \). \( f' < 0 \) for \( x < 0 \) and \( f' > 0 \) for \( x > 0 \). So \( x = 0 \) is a local minimum.

- (c) [10 points] How many different inflection points does the graph \( y = f(x) \) have if \( a < 0 \)? Explain.

\[ \text{Solution:} \quad \text{When } a < 0, f''(x) = 12x^2 + 4a = 0 \text{ if } x = \pm \sqrt{-a/3}. \] The sign of \( f'' \) changes at each, so there are two inflection points in this case.

8. [15 points] A cubical block of dry ice (solid \( CO_2 \)) is evaporating and losing volume at the rate of 10 cm\(^3\)/min. How fast are the sides of cube shrinking when the block has volume 125 cm\(^3\)? Give the units of your answer.

\[ \text{Solution:} \quad \text{Let } x \text{ denote the side of the cube, so the volume is } V = x^3. \] Differentiating with respect to \( t \), we get \( \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \). We know \( \frac{dV}{dt} = -10 \) when \( V = 125 \). At that time, \( x = (125)^{1/3} = 5 \), so

\[ \frac{dx}{dt} = \frac{-10}{3 \cdot 5^2} = \frac{-2}{15} \approx -0.13 \]

(units: cm/min).
9. [20 points] Ship A travels along the path given by the parametric curve \( x = t, y = t^2 \). At the same time, ship B travels along the curve \( x = t, y = 4t - 5 \). At what time are the two ships closest to one another?

**Solution:** At each time \( t \), we can find the distance between the ships using the distance formula for points in the plane:

\[
d(t) = \sqrt{(x_A(t) - y_A(t))^2 + (y_A(t) - y_B(t))^2} = \sqrt{(t - t)^2 + (t^2 - 4t + 5)^2} = |t^2 - 4t + 5|
\]

But note that \( t^2 - 4t + 5 = (t - 2)^2 + 1 > 0 \) for all \( t \). So, we get \( d(t) = t^2 - 4t + 5 \). We want the time when the ships are closest, so we are looking for the minimum of \( d(t) \). To find this, take \( d'(t) = 2t - 4 \) and set \( = 0 \). This says \( t = 2 \).

**Note:** Many people made tables of the distance at whole number values for \( t \): \( t = 0, 1, 2, 3, \) etc., and concluded that the minimum distance occurred at \( t = 2 \) from that information. While the conclusion is correct, the reasoning is not. How do you know that the distance doesn’t reach an even smaller value for some \( t \) between 1 and 2, or \( t \) between 2 and 3? So this solution did not receive full credit, even though the answer was correct.

10.

- (a) [7.5 points] Compute the left- and right-hand sums for the function \( f(t) = e^{-t^2} \) on the interval \([0, 1]\), using \( n = 4 \) equal subdivisions.

**Solution:** \( \Delta t = \frac{1-0}{4} = .25 \).

\[
LHS = e^0 \Delta t + e^{-(25)^2} \Delta t + e^{-(5)^2} \Delta t + e^{-(.75)^2} \Delta t \pm .82
\]

and

\[
LHS = e^{-(25)^2} \Delta t + e^{-(5)^2} \Delta t + e^{-(.75)^2} \Delta t + e^{-1} \Delta t \pm .66
\]

**Note:** A lot of people were computing values of this function incorrectly. I couldn’t tell exactly what went wrong, but I think it was probably a calculator issue. So I didn’t assess a large penalty if your method was OK, but the values were incorrect.

- (b) [7.5 points] Consider the graph \( y = f(t) \) given below. Compute the average value \( \overline{y} \) of \( f \) on the interval \( 0 \leq t \leq 5 \), given that the area marked \( A \) is 25 square units, and the areas marked \( B \) and \( C \) are both 16 square units.
Solution: Since the areas $A$ and $C$ are below the $t$-axis, in computing $\int_0^5 f(t) \, dt$, we count them with negative signs. So the average value is

$$\bar{y} = \frac{1}{5 - 0} \int_0^5 f(t) \, dt = \frac{1}{5}(-25 + 16 - 16) = -5.$$