Comments and Solutions

3J: Since $h(z) = \pi^2 \csc^2(\pi z) - \sum_{n=-\infty}^{\infty}(z-n)^{-2}$, you also need to show that the first term is periodic with period π. This comes from the trig identity (addition formula) $\sin(z+w) = \sin(z)\cos(w) + \sin(w)\cos(z)$. So

$$\sin(\pi z + \pi) = \sin(\pi z)\cos(\pi) + \sin(\pi)\cos(\pi z) = -\sin(\pi z).$$

Hence

$$\pi \csc^2(\pi z + \pi) = \frac{\pi}{\sin^2(\pi z + \pi)} = \frac{\pi}{(-\sin(\pi z))^2} = \pi \csc^2(\pi z).$$

Then, you also need to show that $h(z/2) + h((z+1)/2) = 4h(z)$. In fact, each of the two parts in the definition of $h(z)$ satisfies the same identity. The infinite series part works like this: substitute $z/2$ and $(z+1)/2$, then rearrange as follows:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\left(\frac{z}{2}-n\right)^{-2} + \sum_{n=-\infty}^{\infty}\left(\frac{z+1}{2}-n\right)^{-2} &= 4\sum_{n=-\infty}^{\infty}(z-2n)^{-2} + 4\sum_{n=-\infty}^{\infty}(z-(2n-1))^{-2} \\ &= 4\sum_{k=-\infty}^{\infty}(z-k)^{-2},\end{aligned}$$

since the first sum gives all the terms for even k and the second sum gives all the terms for odd k. The series can be rearranged to get this since they are absolutely convergent on all compact subsets of $\mathbf{C}$.

The other term satisfies the same identity because of trigonometric identities, including the addition formula for the sin function again. We have

$$\begin{aligned}\pi^2\csc^2(\pi z/2) + \pi^2\csc^2(\pi(z+1)/2) &= \frac{\pi^2}{\sin^2(\pi z/2)} + \frac{\pi^2}{\sin^2(\pi z/2 + \pi/2)} \\ &= \frac{\pi^2}{\sin^2(\pi z/2)} + \frac{\pi^2}{\cos^2(\pi z/2)} \\ &= \frac{\pi^2(\cos^2(\pi z) + \sin^2(\pi z))}{(\sin(\pi z/2)\cos(\pi z/2))^2} \\ &= \frac{\pi^2}{\sin^2(\pi z)/4} \\ &= \frac{4\pi^2}{\sin^2(\pi z)} \\ &= 4\pi^2\csc^2(\pi z).\end{aligned}$$

3P: You showed correctly that $\wp'(z)$ and

$$f(z) = \frac{\sigma(z-\omega_1/2)\sigma(z-\omega_2/2)\sigma(z-\omega_3/2)}{\sigma(\omega_1/2)\sigma(\omega_2/2)\sigma(\omega_3/2)\sigma^3(z)}$$

have the same poles and zeroes with the same orders. But then you need to think about the Laurent series expansions of the two sides at $z = 0$ to see the correct constant c so that $\wp'(z) = cf(z)$. You can do that as follows. Since $\sigma(z)$ has a single zero at 0 in the usual period parallelogram, and

$$\sigma(z) = z \prod_{\omega \in \Omega} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

we can write $\sigma(z) = zh(z)$ where $h(z)$ is holomorphic for all z. Therefore, $\sigma'(z) = zh'(z) + h(z)$ and we have $\sigma'(0) = 0 + h(0) = 1$. Hence

$$\sigma(z) = z + \text{h.o.t.}$$

(higher order terms). Similarly, expanding at $z = 0$, and using the fact that σ is an odd function,

$$\sigma(z - \omega_i/2) = \sigma(-\omega_i/2) + \text{h.o.t.} = -\sigma(\omega_i/2) + \text{h.o.t.}.$$

Hence

$$f(z) = \frac{-1 + \text{h.o.t}}{z^3(1 + \text{h.o.t})^3} = \frac{-1}{z^3} + \text{h.o.t.}.$$

Comparing this with the Laurent expansion of $\wp'(z)$:

$$\wp'(z) = \frac{-2}{z^3} + \text{h.o.t.}$$

we see that $c = 2$.

3S: For this you need to use 3.14.3 and 3.14.4. The function $\zeta(z - b)$ has a simple pole at b with residue 1, but it's not elliptic (i.e. not Ω-periodic). However, 3.14.3 says that a sum $\sum_i c_i \zeta(z - b_i)$ is elliptic as long as $\sum_i c_i = 0$. Hence using that and the rest of 3.14.4, an appropriate function is

$$f(z) = \zeta(z - b_1) - \zeta(z - b_2) + 2\wp(z - b).$$

3U: I think you misread the question here. The first part: "Prove that

$$\wp(u) - \wp(v) = \frac{\sigma(v - u)\sigma(v + u)}{\sigma^2(u)\sigma^2(v)}."$$

is a separate statement that needs to be done first. (The other parts follow from that; they don't prove this part.) The way to show this is pretty similar to 3P above. The idea is that if you fix $v \neq 0$ and think of u as varying, then the left side $\wp(u) - \wp(v)$ has a pole of order 2 at $u = 0$ and zeros at $u = \pm v$ (recall that $\wp$ is even). The right side has zeroes at $u = \pm v$ and a pole of order 2 at $u = 0$ as well. Moreover the Laurent expansions of both sides at $u = 0$ start $\frac{1}{u^2}$ + h.o.t, so this time the constant is 1.

Lisa,

I'm not sure what happened here, but this was not your best work. You were missing several big chunks of what were not very difficult problems here. I'm guessing you might have been "crunched for time" due to other things the past week. In any case, your course grade for independent study will be a B+. If you have any questions, please contact me.

John Little