Mathematics 244, section 1 – Linear Algebra
Exam 3 Sample Questions – April 20, 2007

Notation

- $P_n(\mathbf{R})$ is the vector space of polynomials of degree $\leq n$.
- $P(\mathbf{R})$ is the vector space of all polynomials (no restriction on degree).
- $M_{m\times n}(\mathbf{R})$ is the vector space of all $m \times n$ matrices.

I. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear mapping whose matrix with respect to the standard basis is

$$[T]_{std\leftarrow std} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \\ 1 & -2 & 1 \end{pmatrix}$$

Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

be bases for $\mathbf{R}^3$.

A) What is the change of coordinates matrix $\mathcal{P}_{\mathcal{C}\leftarrow std}$?
B) What is the matrix $[T]_{std\leftarrow \mathcal{B}}$?
C) What is the matrix $[T]_{\mathcal{C}\leftarrow std}$?
D) What is the matrix $[T]_{\mathcal{B}\leftarrow \mathcal{C}}$?

II. Let V be a finite-dimenional vector space, and let $T : V \to V$ be a linear mapping. Show by mathematical induction that if $\mathbf{v}_i \in V$ are vectors satisfying $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for $i = 1, \ldots, n$, where λ_i are distinct real numbers, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

III. Let

$$A = \begin{pmatrix} a & -1 & 0 \\ 0 & 3 & 0 \\ 2 & 2 & -4 \end{pmatrix}$$

A) Find the characteristic polynomial of A.
B) Does $\dim\mathrm{Nul}(A - (-4)I)$ depend on the value of $a \in \mathbf{R}$? Explain.
C) For which values of $a \in \mathbf{R}$ is the matrix A diagonalizable?

IV. Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be the mapping defined by

$$T(p(x)) = 2xp'(x) + (x+1)p''(x).$$

A) Show that T is linear.
B) Find the matrix $[T]_{B\leftarrow B}$ with respect to the basis $B = \{1, x, x^2\}$ of $P_2(\mathbf{R})$
C) Determine the the eigenvalues of T.
D) For each eigenvalue, find a basis for the corresponding eigenspace (express the basis elements as polynomials). Is T diagonalizable? Why or why not?

V. (True-False) For each true statement, give a short proof. For each false statement, give a reason or a counterexample.

A) If A is an $n \times n$ matrix whose rows are linearly dependent, then $\lambda = 0$ is an eigenvalue of A.

B) If A is an 8×8 matrix with $\det(A - \lambda I) = (2-\lambda)^3(3-\lambda)^5$, $\text{rank}(A - 2I) = 6$, and $\text{rank}(A - 3I) = 4$, then it is possible for A to be diagonalizable.

C) If the columns of an $n \times n$ matrix Q form an orthogonal set in $\mathbf{R}^n$, then $Q^tQ = I$, then $n \times n$ identity matrix.

D) If S is an orthogonal set of nonzero vectors in $\mathbf{R}^n$ then S is linearly independent.

E) If $\mathcal{B}$ is a linearly independent set in the vector space $M_{3\times 4}(\mathbf{R})$, then $\mathcal{B}$ contains at most $3 + 4 = 7$ matrices.