Mathematics 244, section 1 – Linear Algebra
Sample Exam 2 – March 16, 2007

I. All parts of this question refer to the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 & -1 \\ 2 & 1 & -1 & 1 & s \\ -1 & -1 & 3 & 2 & -1 \end{pmatrix}$$

A) For which value of the scalar s does A satisfy $\mathrm{rank}(A) = 2$? In the next two parts use this value of s.
B) Give a basis of $\mathrm{Col}(A)$.
C) Find the dimension of $\mathrm{Nul}(A)$ and give a basis of $\mathrm{Nul}(A)$.

II.
A) Use Cramer's Rule to solve

$$\begin{aligned} 5x_1 + 3x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 2 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

B) Prove Cramer's Rule: If A is an invertible $n \times n$ matrix, then the solution of $Ax = b$ is the vector x with entries

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

for $i = 1, \ldots, n$.

III. Let $V = P_3(\mathbf{R})$, the vector space of all polynomials of degree ≤ 3 with real coefficients.
A) Is $T : V \to V$ defined by $T(p) = p' - 2p$ a linear transformation (p' = derivative of p)? Why or why not?
B) Is

$$W = \{p \in V : p'(x) - 2p(x) = 0 \text{ for all } x\}$$

a vector subspace of V? Why or why not?

IV. (True-False) For each true statement, give a short proof. For each false statement, give a counterexample. (Note: As always, "true" means "true in every case.")
A) If $T : \mathbf{R}^5 \to \mathbf{R}^3$ is linear, A is the standard matrix of T, and $S = \{v_1, v_2, v_3\} \subset \mathrm{Nul}(A)$, then S is linearly dependent.
B) Let A, B be $n \times n$ matrices. If $\det(A) = -3$ and $B^2 = I$, then $B^t AB$ is an invertible matrix.
C) If A is an $m \times n$ matrix, then $\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = n$.