Mathematics 244 – Linear Algebra, section 1
Problem Set 10 – **Due**: Wednesday, May 2, no later than 5:00pm

I. For each of the following symmetric matrices of A,
(i) Determine the eigenvalues
(ii) Find an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of the matrix
(iii) Give the spectral decomposition of A

A)

$$A = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}$$

B)

$$A = \begin{pmatrix} 16 & -4 \\ -4 & 1 \end{pmatrix}$$

C)

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

II. Let $A \in M_{n\times n}(\mathbf{R})$, with entries a_{ij}, and let $p(\lambda) = \det(A - \lambda I)$ be its characteristic polynomial.
A) Show by induction on n that

$$p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdot \cdots \cdot (a_{nn} - \lambda) + (\text{terms of degree } \leq n-2)$$

(where degree means degree as a polynomial in the variable λ).
B) Deduce from part A that the coefficient of λ^{n-1} in $p(\lambda)$ is $(-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn})$. This scalar, the sum of the diagonal entries in A, is called the *trace* of A and written $\mathrm{Tr}(A)$.
C) Show that if A has n real eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct), then $\mathrm{Tr}(A) = \lambda_1 + \cdots + \lambda_n$ and $\det(A) = \lambda_1 \cdot \cdots \cdot \lambda_n$.

III. An $n \times n$ matrix Q is said to be *orthogonal* if Q is invertible and $Q^{-1} = Q^t$.
A) Show that if Q is orthogonal, then the columns of Q form an orthonormal basis of $\mathbf{R}^n$.
B) Show conversely that if the columns of Q form an orthonormal basis for $\mathbf{R}^n$, then Q is orthogonal.
C) Show that if P and Q are orthogonal, then PQ is also orthogonal.
D) Show that if Q is orthogonal, then Q^{-1} is also orthogonal.