Mathematics 244, section 1 – Linear Algebra
Exam 3 Solutions – April 30, 2007

I. Let $V = \text{Span}\{e^x, \cos(x), \sin(x)\}$, let $W = P_2(\mathbf{R})$ and let $T : V \to W$ be the mapping defined by

$$T(f) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

A) (10) Show that T is a linear mapping.

Solution: Let $f, g \in V$. Then

$$\begin{aligned} T(f+g) &= (f+g)(0) + (f+g)'(0)x + \frac{(f+g)''(0)}{2}x^2 \\ &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + g(0) + g'(0)x + \frac{g''(0)}{2}x^2 \\ &= T(f) + T(g). \end{aligned}$$

Similarly, for $c \in \mathbf{R}$,

$$\begin{aligned} T(cf) &= (cf)(0) + (cf)'(0)x + \frac{(cf)''(0)}{2}x^2 \\ &= cf(0) + cf'(0)x + c\frac{f''(0)}{2}x^2 \\ &= cT(f). \end{aligned}$$

B) (15) Let $B = \{e^x, \cos(x), \sin(x)\}$ (a basis for V) and $C = \{1, x, x^2\}$ (basis for W). Find the matrix $[T]_{C \leftarrow B}$.

Solution: Note that $T(f)$ is just the second-degree Taylor polynomial of f at $a = 0$. So we can compute $T(f)$ using facts from calculus:

$$\begin{aligned} T(e^x) &= 1 + x + \frac{x^2}{2} = 1 \cdot 1 + 1 \cdot x + \frac{1}{2} \cdot x^2 \\ T(\cos(x)) &= 1 - \frac{x^2}{2} = 1 \cdot 1 + 0 \cdot x + \frac{-1}{2} \cdot x^2 \\ T(\sin(x)) &= x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \end{aligned}$$

Hence placing the coordinate vectors of these $T(f)$ into the columns,

$$[T]_{C \leftarrow B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix}.$$

C) (15) Now, let $C' = \{x^2 + x, x - 2, x + 1\}$. Find the matrix $[T]_{C' \leftarrow B}$.

Solution: We have

$$[T]_{C'\leftarrow B} = P_{C'\leftarrow C}[T]_{C\leftarrow B}.$$

The matrix

$$P_{C'\leftarrow C} = (P_{C\leftarrow C'})^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1}.$$

Computing the inverse,

$$\left(\begin{array}{ccc|ccc} 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 3/2 & 1/2 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/3 & 1/3 & -1/3 \\ 0 & 0 & 1 & 1/3 & 2/3 & -2/3 \end{array}\right)$$

So

$$\begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & -2/3 \end{pmatrix}.$$

Then

$$[T]_{C'\leftarrow B} = \begin{pmatrix} 0 & 0 & 1 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & -2/3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1/2 & -1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/6 & -1/6 & 1/3 \\ 2/3 & 2/3 & 2/3 \end{pmatrix}.$$

II. Let

$$A = \begin{pmatrix} 1 & s & s \\ -1 & s & s \\ -1 & -1 & 1 \end{pmatrix}$$

The characteristic polynomial of A is

$$p(\lambda) = -\lambda^3 + (s+2)\lambda^2 - (5s+1)\lambda + 4s.$$

A) (10) Determine a value of s such that $\lambda = 3$ will be an eigenvalue of A.

Solution: This will be the case when

$$0 = -27 + 9(s+2) - 3(5s+1) + 4s = -2s - 12,$$

so $s = -6$.

B) (20) Let $s = -1$. Is A diagonalizable in this case? If so, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. If not, say why not. *Note:* You may "purchase" a factored form of the characteristic polynomial with $s = -1$ from me during the exam for a "price" of 5 points on this part.

Solution: When $s = -1$, the characteristic polynomial is

$$-\lambda^3 + \lambda^2 + 4\lambda - 4 = (\lambda - 1)(4 - \lambda^2) = (\lambda - 1)(2 - \lambda)(2 + \lambda).$$

The eigenvalues are $\lambda = 1, 2, -2$. These are distinct, so we know A is diagonalizable. (Note this also follows from the Spectral Theorem, since A is symmetric when $s = -1$(!)) We have for $\lambda = 1$,

$$\text{Nul}(A - I) = \text{Nul}\begin{pmatrix} 0 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

For $\lambda = 2$,

$$\text{Nul}(A - 2I) = \text{Nul}\begin{pmatrix} -1 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & -1 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Finally, for $\lambda = -2$,

$$\text{Nul}(A + 2I) = \text{Nul}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}.$$

So

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

III. (True/False) For each true statement, give a short proof. For each false statement, give a counterexample. Do any *three* parts. If you submit solutions for all four, I will count any additional points you earn as extra credit.

A) (10) Let A be an $n \times n$ matrix. If $\{v_1, \ldots, v_k\}$ are eigenvectors with distinct eigenvalues λ_i, $i = 1, \ldots, k$, then $\{v_1, \ldots, v_k\}$ is an orthogonal set.

Solution: This is *False*. A counterexample: Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. Then $\lambda = 1, 2$ are eigenvalues, with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. But $v_1 \cdot v_2 = 1 \neq 0$. (The statement is true if A is symmetric, though.)

B) (10) If A and B are $n \times n$ matrices with the same characteristic polynomial, then A and B are similar.

Solution: This is *False*. A counterexample: Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Both A and B have characteristic polynomial $(\lambda - 1)^2$. But A is not diagonalizable while B is diagonalizable (diagonal, even). Hence they cannot be similar. (The converse of this statement is true.)

C) (10) If v is a nonzero vector in $\mathbf{R}^n$, and P_v is the orthogonal projection mapping to the line spanned by v, then the matrix $[P_v]_{\mathcal{B} \leftarrow \mathcal{B}}$ has rank 1 no matter which basis $\mathcal{B}$ is used in $\mathbf{R}^n$.

Solution: This is *True*. $P_v(w) = \frac{v \cdot w}{v \cdot v} v$ is a multiple of v for all w. Hence if $\mathcal{B} = \{b_1, \ldots, b_n\}$ is any basis, then for all i, $P_v(b_i)$ are scalar multiples of v. Hence $[P_v(b_i)]_{\mathcal{B}}$ will all be scalar multiples of $[v]_{\mathcal{B}}$. It follows that the dimension of the column space of $[P_v]_{\mathcal{B} \leftarrow \mathcal{B}}$ is 1 for all bases $\mathcal{B}$.

D) (10) If a 6×6 matrix A has characteristic polynomial $(1-\lambda)^3(6-\lambda)^3$, and $\operatorname{rank}(A - I) = 4$ and $\operatorname{rank}(A - 6I) = 3$, then A is diagonalizable.

Solution: This is *False*. Since $\operatorname{rank}(A - I) = 4$, $\dim \operatorname{Nul}(A - I) = 2$. Similarly, since $\operatorname{rank}(A - 6I) = 3$, $\dim \operatorname{Nul}(A - 6I) = 3$. Since $2 + 3 < 6$, A cannot be diagonalizable.