Mathematics 244 – Linear Algebra
Discussion 3 – Discrete Dynamical Systems
April 16, 2007

Background

The eigenvalues and eigenvectors of the matrix A can be used to study the long-term behavior of discrete-time dynamical systems of the form

$$x_{k+1} = Ax_k. \tag{1}$$

Here $x_k \in \mathbf{R}^n$ represents the state of the system at time $t = k$, $A \in M_{n\times n}(\mathbf{R})$ is the "stage matrix" that connects the state of the system at time $t = k$ to the state at time $t = k+1$. Note that the stage matrix A is a matrix of constants – it does not depend on the time. It would also be possible to consider more general systems

$$x_{k+1} = A_k x_k,$$

in which the way things are changing also changes from time to time. But we will stick to systems of this form (1).

(Comment: A special class, in which the sum of the entries in each column of A is equal to 1, are often called *Markov chains*. Tanya Leise showed in her colloquium talk on Google that he stage matrix A in a Markov chain always has $\lambda = 1$ as an eigenvalue. This fact was a key step in the analysis of the Google "importance vector." Also see section 4.9 in our text for more about this special case.)

In the discussion questions I,II,III below, you will study the systems with $n = 2$ of the form

$$x_{k+1} = \begin{pmatrix} .4 & .3 \\ -p & 1.2 \end{pmatrix} x_k \tag{2}$$

for various values of p. Here $x_k = \begin{pmatrix} O_k \\ S_k \end{pmatrix}$ derives from our in-class example of the interactions of predators (spotted owls) and prey (squirrels). The components of x_k represent the populations of owls and squirrels at time $t = k$.

Discussion Questions

I. Suppose $p = .5$.

A) What are the eigenvalues and eigenvectors of A (decimal approximations are OK here). Give a basis of $\mathbf{R}^2$ consisting of eigenvectors.

B) Suppose the initial populations are $x_0 = \begin{pmatrix} O_0 \\ S_0 \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \end{pmatrix}$. Express x_0 as a linear combination of the eigenvector basis you found in part A, and give a formula for x_k for general k based on this. What happens to the populations "in the long run?"

II.

A) What is the *largest* value of p for which the matrix A has two *real* eigenvalues?

B) Say $p = 1$ now. The method we used in question I breaks down if there are no real eigenvectors. However, we can still compute x_k, plot them, and try to understand the behavior geometrically. Starting from $x_0 = \begin{pmatrix} 10 \\ 30 \end{pmatrix}$ again, compute $x_1, x_2, \ldots, x_{10}$ and plot them carefully (e.g. on graph paper or using Maple). What is your "best guess" about the long-term behavior here?

III. The system (2) is only realistic as a mathematical model of a predator-prey system when $p > 0$. (This is because the presence of the predators should have a *negative effect* on the population of the prey – they are getting eaten, after all!) Nevertheless, we can still look at dynamical systems of the form (2) from the mathematical point of view for all real values of p. And the systems with $p \leq 0$ could model other types of real-world systems.

A) What are the eigenvalues of A when $p = -1.6$? Find an eigenvector for each, and give a formula for x_k based on an expansion of x_0 in terms of the eigenvalues (like what you did in part I B).

B) Same question as in part A, but taking $p = -2$.

IV. For a general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the scalar $a + d$ is called the *trace* of A, writen $\text{Tr}(A)$. The scalar $ad - bc$ is $\det(A)$.

A) Write the characteristic polynomial of A using $\text{Tr}(A)$ and $\det(A)$ to express the coefficients.

B) Using the characteristic polynomial of A:

1) When are both eigenvalues of A real and distinct and of the same sign?
2) When are both eigenvalues of A real and distinct but of opposite signs?
3) When does A have a repeated (real) eigenvalue?
4) When does $\det(A - \lambda I) = 0$ have no real roots (that is, A has no real eigenvalues)?

Express your answers using equalities/inequalities involving $\det(A)$ and $\text{Tr}(A)$. Hint: Think quadratic formula(!)

C) Each of the conditions from part B corresponds to a subset of the $(\text{Tr}, \det)$-plane, where we can "plot a matrix" A according to the values of its trace and determinant. (For example, the matrix $A = \begin{pmatrix} 3 & -2 \\ 2 & -4 \end{pmatrix}$ has $\text{Tr}(A) = 3 - 4 = -1$ and $\det(A) = -8$. We would represent it as the point $(-1, -8)$ in the $(\text{Tr}, \det)$ plane.) Draw and shade the regions corresponding to each condition from part B.

Assignment

Group write-ups due: Monday, April 23.