Mathematics 244 – Linear Algebra
Discussion 2 – Change of Basis, Compositions
April 2, 2007; writeups due: April 4, 2007

Background

We have discussed the matrices of linear mappings and change of basis in general. In this discussion, we will practice using these ideas. We will also introduce compositions of linear mappings and think about their matrices.

Discussion Questions

I. Let $V = P_2(\mathbf{R})$, let $W = M_{2\times 2}(\mathbf{R})$, and consider the mapping $T : V \to W$ defined for $p \in P_2(\mathbf{R})$ by

$$T(p) = \begin{pmatrix} p(0) & p(1) \\ p'(0) & p'(1) \end{pmatrix}.$$

A) Show that T is a linear mapping.

B) Suppose we use the (ordered) bases $B = \{1, x, x^2\}$ in V and $C = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ in W. What is the matrix $[T]_{C\leftarrow B}$?

C) Now suppose we change basis to $B' = \{1 - x^2, x + x^2, 2 + x^2\}$ for $P_2(\mathbf{R})$. What is the matrix $[T]_{C\leftarrow B'}$?

D) Show that $C' = \{e_{11}, e_{22}, e_{12} + e_{21}, e_{12} - e_{21}\}$ is also a basis for W.

E) What is the matrix $[T]_{C'\leftarrow B'}$ (B' same as in part C)?

II. Let V, W, U be three finite-dimensional vector spaces, and let A, B, C be bases in V, W, U respectively.

A) If $T : V \to W$ and $S : W \to U$ are linear mappings, show that the composition $S \circ T : V \to U$ (the mapping defined by $(S \circ T)(v) = S(T(v))$ for all $v \in V$) is also a linear mapping.

B) For example, let $V = P_3(\mathbf{R})$, let $W = P_4(\mathbf{R})$, and let $U = P_4(\mathbf{R})$. Let $B = \{1, x, x^2, x^3, x^4\}$ and $A = C = \{1, x, x^2, x^3\}$ be bases. Let $T : V \to W$ be the mapping $T(p) = P$, where P is the antiderivative of p with $P(0) = 0$. Let $S : W \to U$ be the mapping $S(p) = p'$ (the derivative). What are the matrices

$$[T]_{B\leftarrow A}, [S]_{C\leftarrow B}, [S \circ T]_{C\leftarrow A}$$

What theorem from calculus is reflected in your answer for the matrix of $S \circ T$?

C) In general (not just in the case considered in part B), how are the matrices $[T]_{B\leftarrow A}$, $[S]_{C\leftarrow B}$ and $[S \circ T]_{C\leftarrow A}$ related?

D) **Extra Credit** Prove your assertion in part C.