Mathematics 244 – Linear Algebra
Group Discussion 1 – The Leontieff Input-Output Model
*February 7 and 9, 2007*

*The Leontieff Model*

Recall that $x \in \mathbf{R}^n$ represents the production vector for a national economy, broken into some number $n$ of sectors. The final consumer demand for outputs from each of the sectors is given by the vector $d$. But each sector uses inputs from other sectors as the inputs to produce its outputs. These inputs are represented by the entries of the $n \times n$ consumption matrix $C$. The basic form of the model is the equation

$$(LM_1) \qquad x = Cx + d.$$

We usually think of the entries of $x$ and $d$ as expressed in monetary units (e.g. billions of dollars for a big economy like the U.S. national economy), but we won't be specific about that in the following.

*Discussion Questions*

I. Consider an economy with 3 sectors – manufacturing ($m$), agriculture ($a$), and services ($s$). For each unit of its output, $m$ uses .1 unit from other companies in its sector, .3 units from $a$, and .3 units from $s$. For each unit of its output, $a$ uses .2 units of its output and .6 units from $m$ and .1 units from $s$. For each unit of its output, $s$ uses .6 units from $m$ and .1 units of its own output, but no output from $a$.

A) What is the consumption matrix $C$ that matches the given information?
B) Determine the production levels needed to satisfy a final demand of 18 units from $a$, but no demand for the other sectors. Hint: When you set up your equations to solve, if you are doing this by hand, it might help to "scale" each row to start to get rid of the decimals.
C) Determine the production levels needed to satisfy a final demand of 18 units from $m$, but no demand for the other sectors.
D) What are the production levels needed to satisfy a final demand of 18 units from $m$, and 18 units from $a$, but no demand for $s$. Hint: You don't need any additional calculations here if you have already done parts B and C. Why not?
E) Is $I_3 - C$ an invertible matrix in this example? If so, find $(I_3 - C)^{-1}$.

II. As we have seen, if $(I_n - C)$ is invertible, then $(LM_1)$ has a unique solution

$$x = (I_n - C)^{-1} d$$

for all $d$. Here is another way to think about what this means. The meaning of the matrix $C$ is that

- to produce $d$ directly, $Cd$ inputs are required (adding over all sectors), but

- those inputs require $C(Cd) = C^2 d$ "second round" inputs, and
- the second round inputs require $C(C^2 d) = C^3 d$ "third round" inputs, etc.
- So if we follow this backward process *ad infinitum*, then the total inputs required will be

$$x = d + Cd + C^2 d + C^3 d + \cdots = (I + C + C^2 + C^3 + \cdots)d$$

where the sum extends to infinity (that is, an *infinite series of matrices*!).

A) The infinite series

$$I + C + C^2 + C^3 + \cdots$$

makes sense if and only if $\lim_{n\to\infty} C^n = 0$ (the zero matrix) and

$$\lim_{n\to\infty} I + C + C^2 + \cdots + C^n = D$$

for some matrix $D$. Show that if these conditions are satisfied, then the matrix $D$ *is* $(I - C)^{-1}$. It is enough, by the definition of matrix inverses, to show $D(I - C) = I = (I - C)D$. Start by looking at

$$(I - C)(I + C + C^2 + \cdots + C^n).$$

B) The discussion in part A is closely related to a topic you have seen before, but in a different class. What topic is this?

IV. In the "full" Leontieff model there is a second equation

$$(LM_2) \qquad p = C^t p + v$$

where $v, p$ are vectors in $\mathbf{R}^n$. The vector $v$ represents the "value added" – its components give the value added per unit input for each sector. The vector $p$ is the price vector, the components are the prices per unit output for all the sectors. The matrix $C^t$ is the *transpose* of $C$ above. The transpose of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^t$ whose rows are the columns of $A$ and whose columns are the rows of $A$. For instance:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

A) Show that if $A, B$ are $n \times n$ matrices, then

$$(AB)^t = B^t A^t.$$

(Hint: use the formula for the entries in a matrix product.)

B) From part A, deduce that if $A$ is invertible, then so is $A^t$ and

$$(A^t)^{-1} = (A^{-1})^t.$$

C) Show using matrix algebra that

$$v^t x = p^t d.$$

You may assume that $I_n - C$ is invertible. The products on either side here are

$$(1 \times n)(n \times 1) = (1 \times 1)$$

matrices, so we can think of the result as a single number. The value we get from each these matrix products is the *gross domestic product* of the economy.

*Assignment*

One writeup from each group, due in class on Monday, February 12.