Mathematics 244, section 1 – Linear Algebra
Final Examination Solutions
May 10, 2007

I. Let

$$A = \begin{pmatrix} -1 & 0 & 2 & 2 & -1 \\ 2 & 1 & -1 & 2 & -1 \\ 1 & a & -5 & -8 & a \end{pmatrix}$$

A) (10) Are there any $a \in \mathbf{R}$ such that columns 1,2,3 of A form a linearly dependent set? If so, find such an a. If not, say why not.

Solution: Performing row operations on the first three columns:

$$\begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ 1 & a & -5 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & a & -3 \end{pmatrix}$$
$$\sim \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -3-3a \end{pmatrix}$$

The first three columns will be linearly dependent if $-3 - 3a = 0$, or $a = -1$. They will be linearly independent for any other $a \in \mathbf{R}$.

B) (10) Are there any $a \in \mathbf{R}$ such that the system of equations $Ax = e_1$ (the first standard basis vector in $\mathbf{R}^3$) is inconsistent? If so, for which a is this true? If not, say why not.

Solution: Do the same row operations on the augmented matrix of the system $Ax = e_1$:

$$\left(\begin{array}{ccccc|c} -1 & 0 & 2 & 2 & -1 & 1 \\ 2 & 1 & -1 & 2 & -1 & 0 \\ 1 & a & -5 & -8 & a & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccccc|c} -1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 3 & 6 & -3 & 2 \\ 0 & a & -3 & -6 & a-1 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccccc|c} -1 & 0 & 2 & 2 & -1 & 1 \\ 0 & 1 & 3 & 6 & -3 & 2 \\ 0 & 0 & -3-3a & -6-6a & 4a-1 & 1-a \end{array}\right)$$

From this we can see that the answer to the question is *No.* The first two columns always contain pivots. If $a \neq -1$, then there is a third pivot in column 3, so there are solutions. If $a = -1$, then there is a third pivot in column 5, since $4(-1)-1 = -5 \neq 0$. Therefore, the system $Ax = e_1$ has solutions for all $a \in \mathbf{R}$.

C) (15) Let $a = 4$ and find a parametrization for $\text{Nul}(A)$.

Solution: Continuing from above with $a = 4$ (but omitting the last column):

$$\begin{pmatrix} -1 & 0 & 2 & 2 & -1 \\ 0 & 1 & 3 & 6 & -3 \\ 0 & 0 & -15 & -30 & 15 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & -2 & -2 & 1 \\ 0 & 1 & 3 & 6 & -3 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix}$$

Hence

$$Nul(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ s \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbf{R} \right\}$$

II.

A) (5) *Define:* The *span* of a set S in a vector space V.

Solution: The span of S is the set of all linear combinations of vectors in S, or

$$\text{Span}(S) = \{c_1 x_1 + \cdots + c_k x_k : x_i \in S, c_i \in \mathbf{R}\}.$$

B) (10) Given: if V is a vector space, S is a finite subset of V with $\text{Span}(S) = V$, and T is any linearly independent subset of V, then $|T| \leq |S|$. Use this fact to show that any two bases of V must have the same number of elements.

Solution: Let B and C be any two bases of V. Since B is linearly independent and C spans V, we have by the given fact $|B| \leq |C|$. On the other hand, since B spans V and C is linearly independent, we have $|B| \geq |C|$. Since both inequalities are true, $|B| = |C|$.

C) (5) The result of part B provides the justification for the definition of what concept connected to vector spaces?

Solution: This is the justification for the definition of the *dimension* of a vector space as the number of vectors in a basis.

III.

A) (15) Solve by Cramer's Rule:

$$\begin{aligned} 3x_1 \qquad\quad + x_3 &= 3 \\ 4x_1 + 3x_2 - x_3 &= 2 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

Solution: Computing determinants, we have

$$\det(A) = 3 \cdot 7 - 0 + 1 \times 4 = 25$$

$$\det(A_1(b)) = \det \begin{pmatrix} 3 & 0 & 1 \\ 2 & 3 & -1 \\ 1 & 1 & 2 \end{pmatrix} = 3 \times 7 + 1(-1) = 20$$

$$\det(A_2(b)) = \det \begin{pmatrix} 3 & 3 & 1 \\ 4 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} = 3 \times 5 - 3 \times 8 + 1 \times 4 = -5$$

$$\det(A_1(b)) = \det \begin{pmatrix} 3 & 0 & 3 \\ 4 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix} = 3 \times 1 + 0 + 3 \times 4 = 15$$

Therefore,

$$\begin{aligned} x_1 &= \frac{\det(A_1(b))}{\det(A)} = \frac{20}{25} = \frac{4}{5} \\ x_2 &= \frac{\det(A_2(b))}{\det(A)} = \frac{-5}{25} = -\frac{1}{5} \\ x_3 &= \frac{\det(A_3(b))}{\det(A)} = \frac{15}{25} = \frac{3}{5}. \end{aligned}$$

(It is easy to check that these are correct, by substituting in the original equations.)

B) (5) Suppose $\det(A) = 3$, $\det(B) = -2$, and $\det(C) = 4$. What is $\det(A^t B^{-1} C^2)$?

Solution: By the properties of determinants,

$$\det(A^t B^{-1} C^2) = \det(A) \det(B)^{-1} (\det(C))^2 = 3 \times \frac{1}{(-2)} \times 4^2 = -24.$$

IV. All parts of this problem refer to the linear mapping $T : P_3(\mathbf{R}) \to \mathbf{R}^3$ defined by

$$T(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}$$

A) (10) Show that the *image* of T is a vector subspace of $\mathbf{R}^3$ *directly* (that is, do not just quote a general theorem).

Solution: The image $\mathrm{Im}(T)$ is the set of $y \in \mathbf{R}^3$ such that $y = T(p)$ for some polynomial $p \in P_3(\mathbf{R})$. Since $T(0) = 0$, we know $0 \in \mathrm{Im}(T)$. Let $y_1, y_2 \in \mathrm{Im}(T)$. Then $y_1 = T(p_1)$ and $y_2 = T(p_2)$ for some p_1, p_2. But then since T is linear:

$$y_1 + y_2 = T(p_1) + T(p_2) = T(p_1 + p_2).$$

This shows $y_1 + y_2 \in \mathrm{Im}(T)$ also. Finally if $y = T(p) \in \mathrm{Im}(T)$ and $c \in \mathbf{R}$, then $cy = cT(p) = T(cp)$. Hence $cy \in \mathrm{Im}(T)$. Therefore, $\mathrm{Im}(T)$ is a subspace of $\mathbf{R}^3$.

B) (10) Find the matrix of T with respect to the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ in the domain and the standard basis in $\mathbf{R}^3$.

Solution: We have

$$T(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad T(x) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad T(x^3) = T(x).$$

Therefore, by definition:

$$[T]_{std \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

C) (15) State the change of coordinates formula and use it to find the matrix of T with respect to the basis $\mathcal{B}$ in the domain and the basis

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

in $\mathbf{R}^3$. For full credit, you must show all details of the computation of any matrix inverses you need.

Solution: The change of coordinates formula says that if $T : V \to W$ is linear, $\mathcal{B}, \mathcal{B}'$ are bases of V, and $\mathcal{C}\mathcal{C}'$ are bases of W, then

$$[T]_{\mathcal{C}' \leftarrow \mathcal{B}'} = P_{\mathcal{C}' \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}} [T]_{\mathcal{B} \leftarrow \mathcal{B}'}.$$

Here we want:

$$\begin{aligned}
[T]_{\mathcal{C}\leftarrow\mathcal{B}} &= P_{\mathcal{C}\leftarrow std}[T]_{std\leftarrow\mathcal{B}} \\
&= (P_{std\leftarrow\mathcal{C}})^{-1}[T]_{std\leftarrow\mathcal{B}} \\
&= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & -1 & 0 & -1 \\ 1/2 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1 \end{pmatrix}
\end{aligned}$$

The inverse matrix is computed as usual by row reducing $(P|I) \sim (I|P^{-1})$; details omitted.

V. (10) Show by mathematical induction: If $A = Q^{-1}BQ$ for some invertible Q, then for all integers $n \geq 1$, $A^n = Q^{-1}B^nQ$.

Solution: The base case of the induction is $n = 1$, and there is nothing to prove. For the induction step, assume $A^k = Q^{-1}B^kQ$. Then using the induction hypothesis,

$$A^{k+1} = A \cdot A^k = (Q^{-1}BQ)(Q^{-1}B^kQ)$$

$$= Q^{-1}B(QQ^{-1})B^kQ = Q^{-1}BB^kQ = Q^{-1}B^{k+1}Q,$$

which is what we wanted to show.

VI. Let A be the symmetric matrix:

$$A = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$$

where a, b are real numbers with $b > a > 0$.

A) (10) Show that A has eigenvalues $\lambda = a + b, a, a - b$ and determine the corresponding eigenspaces.

Solution: We have

$$\det A - \lambda I = \det \begin{pmatrix} a - \lambda & 0 & b \\ 0 & a - \lambda & 0 \\ b & 0 & a - \lambda \end{pmatrix}.$$

Expanding along row 2, we have

$$= (a-\lambda)\det\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = (a-\lambda)(\lambda^2 - 2a\lambda + (a^2 - b^2)).$$

This factors easily as

$$= (a-\lambda)((a+b)-\lambda)((a-b)-\lambda).$$

Hence the eigenvalues are $\lambda = a, a+b, a-b$. Under the assumption $b > a > 0$, all three eigenvalues are distinct. We have

$$\mathrm{Nul}(A - aI) = \mathrm{Nul}\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$\mathrm{Nul}(A - (a+b)I) = \mathrm{Nul}\begin{pmatrix} -b & 0 & b \\ 0 & -b & 0 \\ b & 0 & -b \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\right\}$$

$$\mathrm{Nul}(A - (a-b)I) = \mathrm{Nul}\begin{pmatrix} b & 0 & b \\ 0 & b & 0 \\ b & 0 & b \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\right\}$$

B) (10) Find the spectral decomposition of A.

Solution: Write u_i for the basis vectors above for the three eigenspaces. Note that we normalized so that $\{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbf{R}^3$ (this is necessary for the next step). The spectral decomposition is $A = au_1u_1^t + (a+b)u_2u_2^t + (a-b)u_3u_3^t =$

$$a \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (a+b)\cdot\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} + (a-b)\cdot\begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}.$$

VII. True - False. For each true statement give a brief proof (saying "this is true by definition" is *not sufficient*!). For each false statement give a counterexample or a reason.

A) (10) If A is an $n \times n$ matrix and $AQ = QD$ for some diagonal matrix D and some matrix Q, then A is diagonalizable.

Solution: This is **False**. The equation $AQ = QD$ does mean that all the columns of Q are eigenvectors of A. But it is not stated that Q is invertible, which means that the columns of Q do not need to form a basis for $\mathbf{R}^n$, and A does not need to be diagonalizable. A counterexample: let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which is not diagonalizable. If $Q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then

$$AQ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} I_2 = QD,$$

where $D = I_2$ is the identity matrix.

B) (10) If $A \in M_{4\times 5}(\mathbf{R})$ and $\dim \mathrm{Col}(A) = 2$, then there are 4 free variables in the system $Ax = 0$.

Solution: This is **False**. A reason is that $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A) = 5$, so if $\dim \mathrm{Col}(A) = 2$, then $\dim \mathrm{Nul}(A) = 3$. Since the set of solutions of $Ax = 0$ is $\mathrm{Nul}(A)$ by definition, there will only be 3 free variables here.

C) (10) If $\{v_1, v_2, v_3\}$ is a linearly independent subset of $\mathbf{R}^3$ and V is the matrix with columns v_1, v_2, v_3, then $\det(V) \neq 0$.

Solution: This is **True**. By one of the parts of the Invertible Matrix Theorem (see p. 129 in the text), V is invertible. Hence $\det(V) \neq 0$.

D) (10) If A is a symmetric matrix and $v_1, v_2, \ldots, v_k$ are eigenvectors of A with distinct corresponding eigenvalues, then $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal set.

Solution: This is **True**. Suppose $Av_i = \lambda_i v_i$ with distinct λ_i. Then for all pairs $i \neq j$,

$$\begin{aligned} \lambda_i(v_i \cdot v_j) &= (\lambda_i v_i) \cdot v_j \\ &= (Av_i) \cdot v_j \\ &= v_i \cdot (Av_j) \quad A \text{ symmetric} \\ &= v_i \cdot (\lambda_j v_j) \\ &= \lambda_j (v_i \cdot v_j) \end{aligned}$$

This shows $0 = (\lambda_i - \lambda_j)(v_i \cdot v_j)$. Since the first factor is not zero, the second one must be. Therefore v_i and v_j are orthogonal. Since this is true for all pairs i, j, the set of eigenvectors is an orthogonal set.

VIII. (The Cayley-Hamilton Theorem.)

A) (10) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any 2×2 matrix. Show that A "satisfies its own characteristic polynomial equation" in the sense that $A^2 - \mathrm{Tr}(A)A + \det(A)I_2 = 0_{2\times 2}$ (here I_2 is the 2×2 identity matrix and $0_{2\times 2}$ is the 2×2 zero matrix).

Solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By direct calculation: $A^2 - \mathrm{Tr}(A)A + \det(A)I_2 =$

$$\begin{aligned}
&\begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\
&= \begin{pmatrix} a^2+bc-a^2-ad+ad-bc & ab+bd-ab-bd \\ ca+dc-ac-dc & cb+d^2-ad-d^2+ad-bc \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Thus the Cayley-Hamilton theorem is true for 2×2 matrices.

B) (10) Prove that an $n \times n$ diagonalizable matrix A satisfies its own characteristic polynomial in the same sense as in part A: If

$$\det(A - tI) = (-1)^n t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0,$$

then

$$(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n = 0_{n\times n}.$$

Caution: It *does not* make sense to "substitute $t = A$" in $\det(A - tI)$" because t is a scalar, not a matrix. The result of problem V above may be useful. (*Comment:* The result is also true without the extra hypothesis of diagonalizability but it is more difficult to prove then.)

Solution: Under the assumption that A is diagonalizable, there is an invertible matrix Q and a diagonal matrix D with diagonal entries λ_i, the eigenvalues of A, such that $A = Q^{-1}DQ$. By problem V above, for all positive integers k, $A^k = Q^{-1}D^kQ$, and the diagonal entries of D^k are just the powers of the eigenvalues λ_i^k. Substituting for the various powers of A, and factoring out Q^{-1} on left and Q on right,

$$\begin{aligned}
&(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n \\
&= (-1)^n Q^{-1}D^nQ + c_{n-1}Q^{-1}D^{n-1}Q + \cdots + c_1 Q^{-1}AQ + c_0 Q^{-1}Q \\
&= Q^{-1}((-1)^n D^n + c_{n-1}D^{n-1} + \cdots + c_1 D + c_0 I_n)Q
\end{aligned}$$

Since D is diagonal, the middle matrix here,

$$(-1)^n D^n + c_{n-1}D^{n-1} + \cdots + c_1 D + c_0 I_n,$$

is also a diagonal matrix. The ith diagonal entry is

$$(-1)^n \lambda_i^n + c_{n-1}\lambda_i^{n-1} + \cdots + c_1\lambda_i + c_0.$$

For all i, this equals 0, since it is $\det(A - \lambda_i I) = 0$. Hence the whole matrix in the middle is the $n \times n$ zero matrix, which means the whole product is the zero matrix.

IX. (Eigenvalues in calculus!) Let $V = C^\infty(\mathbf{R})$ be the vector space of all functions $f : \mathbf{R} \to \mathbf{R}$ that have derivatives of all orders at all $x \in \mathbf{R}$.

A) (10) Let $D : V \to V$ be the linear mapping defined by $D(f) = f'$. Show that for all $\lambda \in \mathbf{R}$, the eigenspace $\mathrm{Ker}(D - \lambda I)$ has dimension 1, and find a basis.

Solution: We have that $\mathrm{Ker}(D - \lambda I)$ equals the set of all functions f satisfying $(D - \lambda I)(f) = 0$. Here I means the identity mapping on V, so $I(f) = f$ for all f. We want to find all functions satisfying:

$$D(f) = f' = \lambda f, \quad \text{or} \quad \frac{df}{dx} = \lambda f.$$

This is just the usual exponential growth/decay differential equation. We can solve it by separation of variables as usual:

$$\begin{aligned} \int \frac{df}{f} &= \int \lambda dx \\ \ln|f| &= \lambda x + k \\ f &= \pm e^k e^{\lambda x} \\ &= ce^{\lambda x}, \end{aligned}$$

where $c \in \mathbf{R}$ is arbitary. Hence the eigenspace is one-dimensional, and $\{e^{\lambda x}\}$ is one basis.

B) (5) Let $S : V \to V$ be the linear mapping defined by $S(f) = f''$. Let $\lambda = -k^2$ for an arbitrary $k > 0$. Find two linearly independent functions in the eigenspace $\mathrm{Ker}(S - \lambda I)$.

Solution: Recall the derivative formulas

$$\frac{d}{dx}\sin(x) = \cos(x); \qquad \frac{d}{dx}\cos(x) = -\sin(x).$$

Thus

$$S(\sin(x)) = \frac{d^2}{dx^2}\sin(x) = -\sin(x); \qquad S(\cos(x)) = \frac{d^2}{dx^2}\cos(x) = -\cos(x).$$

By the chain rule for derivatives,

$$\{\sin(kx), \cos(kx)\}$$

satisfy

$$S(\sin(kx)) = \frac{d^2}{dx^2}\sin(kx) = -k^2\sin(kx); \qquad S(\cos(kx)) = \frac{d^2}{dx^2}\cos(kx) = -k^2\cos(kx).$$

So $\{\sin(kx), \cos(kx)\}$ is one such set of functions.

C) (5) Let $S : V \to V$ be the linear mapping defined by

$$S(f) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2.$$

Show that $\text{Span}\{1, x, x^2\} \subseteq \text{Ker}(S - I)$. For $f(x) = x^2$,

$$\begin{aligned} T(f) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \\ &= 1 + 2\cdot(x-1) + \frac{2}{2}(x-1)^2 \\ &= 1 + 2x - 2 + x^2 - 2x + 1 \\ &= x^2 \end{aligned}$$

Thus $T(f) = f$ and f is an eigenfunction of T with eigenvalue 1. The proofs for $f = 1, x$ are similar, and easier. Since $\text{Ker}(S - I)$ is a subspace of V, this shows $\text{Span}\{1, x, x^2\} \subseteq \text{Ker}(S - I)$. (Note what this says: the second degree Taylor polynomial of a polynomial of degree ≤ 2 at $a = 1$ *equals the polynomial itself*. The same thing is true for all a, and all degrees! This also shows that the inclusion is an equality!)