Mathematics 244, section 1 – Linear Algebra
Solutions for Final Exam Review Questions
May 4, 2007

I. The following matrix is the augmented matrix for a system of 3 linear equations in 4 unknowns. For which value(s) of $a \in \mathbf{R}$ does the system have a solution? Find all solutions for those a.

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & 1 & 2 \\ 3 & 1 & 3 & 0 & 0 \\ 1 & 12 & 6 & 5 & a \end{array}\right)$$

Solution: Applying row operations,

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & 1 & 2 \\ 3 & 1 & 3 & 0 & 0 \\ 1 & 12 & 6 & 5 & a \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -2 \\ 0 & 7 & 3 & 3 & 6 \\ 0 & 14 & 6 & 6 & a+2 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -2 \\ 0 & 7 & 3 & 3 & 6 \\ 0 & 0 & 0 & 0 & a-10 \end{array}\right)$$

The system is inconsistent unless $a = 10$. When $a = 10$, we continue reducing to row-reduced echelon form:

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 6/7 & -1/7 & -2/7 \\ 0 & 1 & 3/7 & 3/7 & 6/7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Here, x_3 and x_4 are free variables. The solutions in parametric form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2/7 \\ 6/7 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -6/7 \\ -3/7 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1/7 \\ -3/7 \\ 0 \\ 1 \end{pmatrix},$$

where x_3 and x_4 are arbitrary real numbers.

II. Let $V = P_4(\mathbf{R})$, and let $W = \{p(x) \in V : p'(1) = 0 \text{ and } p(0) = 3p(2)\}$.
A) Show that W is a vector subspace of V.

Solution: Let $p, q \in W$, so $p'(1) = q'(1) = 0$ and $p(0) = 3p(2)$, $q(0) = 3q(2)$. Then

$$(p+q)'(1) = p'(1) + q'(1) = 0 + 0 = 0,$$

and

$$(p+q)(0) = p(0) + q(0) = 3p(2) + 3q(2) = 3(p+q)(2).$$

Hence W is closed under sums. In addition, if $r \in \mathbf{R}$,

$$(rp)'(1) = rp'(1) = r \cdot 0 = 0$$

and

$$(rp)(0) = r \cdot p(0) = r \cdot 3p(2) = 3(rp)(2).$$

Hence W is also closed under scalar multiples, and a subspace of V.

B) Find a subset $S \subset W$ such that $W = \mathrm{Span}(S)$.

Solution: Write $p \in V$ as $p = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. Then the equations $p'(1) = 0$ and $p(0) = 3p(2)$ say

$$\begin{aligned} 0 &= a_1 + 2a_2 + 3a_3 + 4a_4 \\ 0 &= 2a_0 + 6a_1 + 12a_2 + 24a_3 + 48a_4 \end{aligned}$$

In matrix form this is the homogenenous system $Aa = 0$ for

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 6 & 12 & 24 & 48 \end{pmatrix}$$

(and $a =$ the vector of coefficients – the coordinate vector of p with respect to the standard basis in V). The row-reduced echelon form is

$$E = \begin{pmatrix} 1 & 0 & 0 & 3 & 12 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

There are three free variables, so the dimension of the space of solutions is three.

$$S = \mathrm{Span}\{-2x + x^2, -3 - 3x + x^3, -12 - 4x + x^4\}$$

is a set that spans W. (Note: these are the polynomial forms of the coordinate vectors spanning the solutions of the echelon form system $Ea = 0$.)

C) Is your set S linearly independent? Justify your assertion.

Solution: Yes it is linearly independent. If

$$0 = c_1(-2x + x^2) + c_2(-3 - 3x + x^3) + c_3(-12 - 4x + x^4),$$

then equating coefficients of x^2 on both sides, $c_1 = 0$. Similarly $c_2 = c_3 = 0$ from the coefficients of x^3 and x^4. (Recall, this is a general property of the spanning set of the null space of a matrix computed from the row-reduced echelon form.)

III. Let V be a vector space. Show that a set $S \subset V$ is linearly dependent if and only if there is some $\mathbf{x} \in S$ such that $\mathbf{x}$ is a linear combination of the vectors in $\mathrm{Span}(S - \{\mathbf{x}\})$.

Solution: See class notes.

IV. All parts of this problem refer to the linear mapping $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (4x_1 - 3x_2 + 2x_3 - x_4, x_2 + x_3, x_1 - x_4)$$

(*Added note:* all vectors here are written as row vectors to save space, but you should read them as column vectors – that is, transpose them – to perform the calculations in the following questions.)

A) Find the matrix of T with respect to the standard bases in the domain and target.

Solution: By the standard recipe

$$\begin{aligned} T(e_1) &= T(1,0,0,0) = (4,0,1) \\ T(e_2) &= T(0,1,0,0) = (-3,1,0) \\ T(e_3) &= T(0,0,1,0) = (2,1,0) \\ T(e_4) &= T(0,0,0,1) = (-1,0,-1) \end{aligned}$$

Hence

$$[T]_{std \leftarrow std} = \begin{pmatrix} 4 & -3 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

B) Find the matrix of T with respect to the basis

$$\mathcal{B} = \{(1,-1,0,0), (1,1,0,0), (0,0,2,3), (0,0,-1,2)\}$$

in the domain and

$$\mathcal{C} = \{(0,0,1), (0,-1,1), (-1,1,0)\}$$

in the target.

Solution: By the change of coordinates formula

$$\begin{aligned} [T]_{\mathcal{C} \leftarrow \mathcal{B}} &= P_{\mathcal{C} \leftarrow std}[T]_{std \leftarrow std} P_{std \leftarrow \mathcal{B}} \\ &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 & -3 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 7 & 1 & 1 & -4 \\ -1 & 1 & 2 & -1 \\ 1 & 1 & -3 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 3 & 0 & -7 \\ -6 & -2 & -3 & 5 \\ -7 & -1 & -1 & 4 \end{pmatrix}. \end{aligned}$$

V. Show by mathematical induction that the determinant of an $n \times n$ upper-triangular matrix $A = (a_{ij})$ is the product of the diagonal entries: $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Solution: The base case is trivial: For a 1×1 matrix

$$\det(a_{11}) = a_{11}$$

by definition. Now assume that the result is true for all $k \times k$ upper-triangular matrices, and consider a $(k+1) \times (k+1)$ upper-triangular matrix A. Expanding along row $k+1$, we see

$$\det(A) = 0 + \cdots + 0 + (-1)^{2k+2} a_{k+1,k+1} \det(A_{k+1,k+1}).$$

The matrix $A_{k+1,k+1}$ is an upper-triangular $k \times k$ matrix. So the induction hypothesis shows that

$$\det(A_{k+1,k+1}) = a_{11} \cdots a_{kk}.$$

Hence

$$\det(A) = a_{k+1,k+1} a_{11} \cdots a_{kk} = a_{11} \cdots a_{kk} a_{k+1,k+1},$$

which is what we wanted to show.

VI. Let

$$A = \begin{pmatrix} 2 & 9 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix}$$

A) Find the eigenvalues of A.

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 9 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & 0 & 5-\lambda \end{pmatrix}$$

Expanding along the third row

$$= (5-\lambda) \det \begin{pmatrix} 2-\lambda & 9 \\ 1 & 2-\lambda \end{pmatrix} = (5-\lambda)(\lambda^2 - 4\lambda - 5) = (5-\lambda)^2(-1-\lambda)$$

The eigenvalues are $\lambda = 5$ (multiplicity 2), and $\lambda = -1$.

B) For each eigenvalue λ, find a basis of $\text{Nul}(A - \lambda I)$.

Solution: For $\lambda = 5$,

$$A - 5I = \begin{pmatrix} -3 & 9 & 0 \\ 1 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

There is only one free variable in the echelon form system. So a basis for $\mathrm{Nul}(A-5I)$ is

$$\left\{\begin{pmatrix}3\\1\\0\end{pmatrix}\right\}.$$

Then, for $\lambda=-1$,

$$A+I=\begin{pmatrix}3&9&0\\1&3&-1\\0&0&6\end{pmatrix}\sim\begin{pmatrix}1&3&0\\0&0&1\\0&0&0\end{pmatrix}$$

There is alson only one free variable in the echelon form system. So a basis for $\mathrm{Nul}(A+I)$ is

$$\left\{\begin{pmatrix}-3\\1\\0\end{pmatrix}\right\}.$$

C) Is A diagonalizable? Explain.

Solution: A is *not* diagonalizable because

$$\dim\mathrm{Nul}(A-5I)+\dim\mathrm{Nul}(A+I)=2$$

(not 3).

VII. True - False. For each true statement give a brief proof. For each false statement give a counterexample or a reason.

A) If $\mathbf{y},\mathbf{z}$ are two fixed vectors in $\mathbf{R}^n$, then the mapping $T:\mathbf{R}^n\to\mathbf{R}^n$ defined by

$$T(\mathbf{x})=(\mathbf{x}\cdot\mathbf{y})\mathbf{y}+(\mathbf{x}\cdot\mathbf{z})\mathbf{z}$$

is linear.

Solution: This is **True**. To prove it, use the fact that the dot product is linear in the first variable.

B) No matter what the entries $a,b,c\in\mathbf{R}$ are, all the roots of the characteristic polynomial of the matrix $A=\begin{pmatrix}a&b\\b&c\end{pmatrix}$ are real.

Solution: This is **True**. For the proof, use the quadratic formula to determine the roots of the characteristic polynomial. Both roots are real because the discriminant (the expression in the square root) is always ≥ 0. This fact was also derived in class – see the notes. (Or, you could also say: This follows from the Spectral Theorem since A is a symmetric matrix.)

C) If $A\in M_{2\times 5}(\mathbf{R})$, then for every $b\in\mathbf{R}^2$ the general solution of the equation $Ax=b$ contains three arbitrary constants.

Solution: This is **False**. The system could be inconsistent (so there are no solutions at all). For instance to make a counterexample, let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and let b be any vector in $\mathbf{R}^2$ whose second component is nonzero. Then b is not in the column space of A, and there are no solutions of $Ax = b$.

D) If A is an $n \times n$ matrix and $\det(A^2 - 49I) = 0$, then there exists a nonzero vector x such that $Ax = 7x$.

Solution: This is **False**. This would say that $\lambda = 7$ must be an eigenvalue of A. But here is a counterexample:

$$A = \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix}.$$

Note that $\lambda = +7$ is not an eigenvalue of A, but $\lambda = 49$ is an eigenvalue of

$$A^2 = \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix}.$$

E) If $A, B \in M_{n\times n}(\mathbf{R})$, and $B = Q^{-1}AQ$ for some invertible $n \times n$ matrix Q, then $\det(B) = \det(A)$.

Solution: This is **True**. For the proof, use standard properties of determinants (the product formula, the inverse formula).

F) If $A \in M_{6\times 9}(\mathbf{R})$ then $\dim \operatorname{Col}(A)$ can equal 8.

Solution: This is **False**. The column space of A is a subspace of $\mathbf{R}^6$. No subspace of $\mathbf{R}^6$ has dimension larger than 6.

G) If $A \in M_{6\times 9}(\mathbf{R})$ then $\dim \operatorname{Nul}(A)$ is at most 6.

Solution: This is also **False**. Counterexample: If A is the 6×9 zero matrix, then $\dim \operatorname{Nul}(A) = 9$ (it's all of $\mathbf{R}^9$).

H) If $A \in M_{6\times 9}(\mathbf{R})$ and $\dim \operatorname{Nul}(A) = 4$, then $\dim \operatorname{Col}(A) = 5$.

Solution: This is **True**. Proof uses the fact that if A is $m \times n$, then

$$\dim \operatorname{Nul}(A) + \dim \operatorname{Col}(A) = n.$$

I) If U is a 7×3 matrix whose columns $\{u_1, u_2, u_3\}$ form an orthonormal set in $\mathbf{R}^7$, then u_1, u_2, u_3 are eigenvectors of the 7×7 matrix UU^t.

Solution: This is **True**. They are all eigenvectors with eigenvalue 1. Because of the orthonormality, $U^t u_1 = e_1$, $U^t u_2 = e_2$ and $U^t u_3 = e_3$ (the standard basis vectors in $\mathbf{R}^3$. Therefore, $UU^t u_i = Ue_i = u_i$ for $i = 1, 2, 3$.

J) If A is a 3×3 matrix with eigenvalues $\lambda = 2, 5, -7$ then Cramer's Rule can be used to solve $Ax = b$ for all $b \in \mathbf{R}^3$.

Solution: This is **True**. The determinant of A is the value of the characteristic polynomial at $\lambda = 0$. Since we know the roots, the polynomial factors as

$$\det(A - \lambda I) = (2 - \lambda)(5 - \lambda)(-7 - \lambda).$$

Therefore $\det(A) = -70 \neq 0$. It follows that A is invertible, so Cramer's Rule can be used to solve $Ax = b$ no matter what b is.

VIII. (*more challenging*)

A) Show that there exists a basis $\{p_1(x), p_2(x), p_3(x)\}$ for $V = P_2(\mathbf{R})$ satisfying

$$\begin{array}{lll} p_1(0) = 1, & p_1(1) = 0, & p_1(2) = 0 \\ p_2(0) = 0, & p_2(1) = 1, & p_2(2) = 0 \\ p_3(0) = 0, & p_3(1) = 0, & p_3(2) = 1 \end{array}$$

Be sure to say how you know your polynomials form a basis for V.

Solution: To get a quadratic polynomial p_1 that satisfies $p_1(1) = p_1(2) = 0$, note that $x = 1, 2$ are the roots. So the polynomial must factor as

$$p_1(x) = c(x - 1)(x - 2)$$

for some constant c. The constant must be chosen to get $p_1(0) = 1$, so $c = 1/2$. Hence $p_1(x)$ can be written as

$$p_1(x) = \frac{(x - 1)(x - 2)}{(0 - 1)(0 - 2)}$$

Similarly, the other two polynomials are

$$p_2(x) = \frac{(x - 0)(x - 2)}{(1 - 0)(1 - 2)}$$

and

$$p_3(x) = \frac{(x - 0)(x - 1)}{(2 - 0)(2 - 1)}$$

To see that these are a basis for $P_2(\mathbf{R})$, note that this vector space has dimension 3 (the standard basis $\{1, x, x^2\}$ has three elements). Hence to show a set of three polynomials in here is a basis, we only have to show that it is linearly independent. If

$$0 = c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x)$$

for all x, then substituting $x = 0$ gives

$$0 = c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0.$$

Hence $c_1 = 0$. Similarly, setting $x = 1$ and $x = 2$ shows $c_2 = c_3 = 0$ also.

B) Suppose you knew that the polynomial $q(x)$ satisfied $q(0) = 8$, $q(1) = -3$ and $q(2) = 4$. What would the expansion of $q(x)$ in terms of the basis $\{p_1(x), p_2(x), p_3(x)\}$ be?

Solution: The expansion is

$$q(x) = 8p_1(x) + (-3)p_2(x) + 4p_3(x).$$

(Note what happens if you substitute $x = 0, 1, 2$ here.)

C) Generalize what you did in part A to show that given any three distinct real numbers a, b, c, there exists a basis $\{p_1(x), p_2(x), p_3(x)\}$ of $V = P_2(\mathbf{R})$ satisfying

$$\begin{array}{lll} p_1(a) = 1, & p_1(b) = 0, & p_1(c) = 0 \\ p_2(a) = 0, & p_2(b) = 1, & p_2(c) = 0 \\ p_3(a) = 0, & p_3(b) = 0, & p_3(c) = 1 \end{array}$$

Solution: Following the recipe from part A, the desired basis is

$$\left\{ \frac{(x-b)(x-c)}{(a-b)(a-c)}, \frac{(x-a)(x-c)}{(b-a)(b-c)}, \frac{(x-a)(x-b)}{(c-a)(c-b)} \right\}$$

Note that a, b, c must be distinct or else we would be dividing by zero, and who knows what would happen then? The paper might burst into flames or something similarly dire! The given equations imply linear independence as in part A. *Added Note:* The $p_i(x)$ are called *Lagrange interpolating polynomials*. If you are taking Numerical Analysis next year, you will use this idea *a lot*(!)

IX. (*more challenging*)

A) Show that if $\mathbf{x}$ is an eigenvector of a matrix A with eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of A^2.

Solution: We have

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}.$$

Hence $\mathbf{x}$ is an eigenvector of A with eigenvalue λ^2.

B) Suppose that A is symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$, and corresponding eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ forming an orthonormal basis of $\mathbf{R}^n$. Let $P_1, \ldots, P_n$ be the matrices of the orthogonal projections onto the lines spanned by the eigenvectors of A. Explain why

$$A^2 = \lambda_1^2 P_1 + \cdots + \lambda_n^2 P_n,$$

using the Spectral Theorem.

Solution: Using the fact that A is symmetric, the matrix A^2 satisfies

$$(A^2)^t = (AA)^t = A^t A^t = AA = A^2.$$

Therefore, A^2 is also symmetric. The Spectral Theorem applied to A says that there is an orthonormal basis $\{u_1, \ldots, u_n\}$ of $\mathbf{R}^n$ consisting of eigenvectors of A. But part A shows that the u_i are also eigenvectors of A^2:

$$Au_i = \lambda_i u_i \Rightarrow A^2 u_i = \lambda_i^2 u_i.$$

The second part of the Spectral Theorem (the spectral decomposition) then shows

$$A^2 = \lambda_1^2 u_1 u_1^t + \cdots + \lambda_n^2 u_n u_n^t.$$

The matrix $P_i = u_i u_i^t$, so this is what we wanted to show.

C) Verify that the formula in part B is true in the case of the 2×2 symmetric matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$.

Solution: The spectral decomposition of A is

$$A = 5 \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Then

$$5^2 \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 3^2 \cdot \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix} = A^2$$

which is what we wanted to show.