Mathematics 244, section 1 – Linear Algebra
Review Sheet, Final Exam
April 30, 2007

General Information

The final examination for this course will be given at 8:30 a.m. on Thursday, May 10 in our regular class room, Swords 359. It will be a comprehensive exam, covering all the material we have studied each semester, divided *roughly* in thirds according to the topics from the three hour exams, plus a question about material related to the Spectral Theorem for real symmetric matrices from the last week of the course. Some questions making use of earlier material (especially reduction to row-reduced echelon form, solving systems of equations, etc.) may appear in the context of topics covered later in the semester. Also, there will be one *more challenging* problem that will ask you to put things you have learned together in new ways.

As a whole, the exam will be roughly twice the length of one of the three midterms, but you will have the full three hour period from 8:30 am to 11:30 am to work on it if you need that much time.

Topics to be Covered

Note: This is not quite the order the topics were presented in class; it shows some logical interconnections, though.

1) Systems of linear equations, row reduction and echelon form, pivot positions, solution sets of systems of linear equations in parametric form, homogeneous and inhomogeneous systems, the connection between the solutions of $Ax = b$ and $Ax = 0$.
2) Matrix operations and matrix inverses
3) Vector spaces, and the key examples: $\mathbf{R}^n$, $P_n(\mathbf{R})$, $M_{m\times n}(\mathbf{R})$, vector spaces of functions, etc.
4) Vector subspaces, including null and column spaces of a matrix
5) Linear combinations and spans
6) Linear independence and dependence for sets of vectors
7) Bases and dimension (know how to find bases for the column and null spaces of a given matrix and determine its rank in particular), coordinate systems in a vector space V, the coordinate vector of a vector with respect to a given basis, change of coordinates matrices.
8) Linear transformations $T : V- > W$ and their matrices, the change of coordinates formula for $[T]$.
9) The determinant of an $n\times n$ matrix, cofactor expansions, properties of determinants (in particular, the effect of the various row operations on determinants, $\det(A) = 0 \Leftrightarrow A$ is not invertible, $\det(AB) = \det(A)\det(B)$), Cramer's Rule for solving $Ax = b$ when $\det(A) \neq 0$, Determinants and areas, volumes.
10) Eigenvalues and eigenvectors, the characteristic equation.
11) Diagonalization and diagonalizability, similarity of matrices.

12) Dot products, orthogonality, orthogonal projections.
13) The Spectral Theorem for real symmetric matrices.

Proofs to Know

See the review sheets for Exams 1, 2, and 3. (These are reposted on the course homepage in case you need additional copies), and:

- Show that if A is a real symmetric matrix then all eigenvalues of A are real, and that eigenvectors for distinct eigenvalues are orthogonal.

Suggestions on How to Study

As I have said before, linear algebra is an *extremely cumulative* subject. But this can and should actually be working *to your benefit now*(!) If you have worked conscientiously at learning the concepts and techniques we have studied, things that seemed confusing earlier just might be "crystal clear" now(!)

To prepare for the final, start by reading the above list of topics carefully. If there are terms there that are unfamiliar or for which you cannot give the precise definition, *learn the definitions now.* Review the class notes. *Most of the problems on the final will be similar to something we have discussed at some point this semester, although things might be stated differently, or appear in a new context.* As mentioned above, there may also be a more challenging question that will require you to put things together in a new way. Also look back over your graded problem sets and exams.

If there are problems that you did not get the first time around, try them again now. Then go through the suggested problems from the review sheets. If you have worked these out previously, it is not necessary to do them all again. But try a representative sample "from scratch" – don't just look over your old solutions and nod your head if it looks familiar. You need the practice thinking through the logic of how the solution is derived again!

Review Session

I will be happy to run a review session for the final exam during study week. We can discuss a time in class on Monday, April 30.

Some Sample Exam Questions

I. The following matrix is the augmented matrix for a system of 3 linear equations in 4 unknowns. For which value(s) of $a \in \mathbf{R}$ does the system have a solution? Find all solutions for those a.

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & 1 & 2 \\ 3 & 1 & 3 & 0 & 0 \\ 1 & 12 & 6 & 5 & a \end{array}\right)$$

II. Let $V = P_4(\mathbf{R})$, and let $W = \{p(x) \in V : p'(1) = 0 \text{ and } p(0) = 3p(2)\}$.
A) Show that W is a vector subspace of V.
B) Find a subset $S \subset W$ such that $W = \text{Span}(S)$.
C) Is your set S linearly independent? Justify your assertion.

III. Let V be a vector space. Show that a set $S \subset V$ is linearly dependent if and only if there is some $\mathbf{x} \in S$ such that $\mathbf{x}$ is a linear combination of the vectors in $\text{Span}(S - \{\mathbf{x}\})$.

IV. All parts of this problem refer to the linear mapping $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (4x_1 - 3x_2 + 2x_3 - x_4, x_2 + x_3, x_1 - x_4)$$

A) Find the matrix of T with respect to the standard bases in the domain and target.
B) (15) Find the matrix of T with respect to the basis

$$\mathcal{B} = \{(1, -1, 0, 0), (1, 1, 0, 0), (0, 0, 2, 3), (0, 0, -1, 2)\}$$

in the domain and

$$\mathcal{C} = \{(0, 0, 1), (0, -1, 1), (-1, 1, 0)\}$$

in the target.

V. Show by mathematical induction that the determinant of an $n \times n$ upper-triangular matrix $A = (a_{ij})$ is the product of the diagonal entries: $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

VI. Let

$$A = \begin{pmatrix} 2 & 9 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix}$$

A) Find the eigenvalues of A.
B) For each eigenvalue λ, find a basis of $\text{Nul}(A - \lambda I)$.
C) Is A diagonalizable? Explain.

VII. True - False. For each true statement give a brief proof. For each false statement give a counterexample or a reason.
A) If $\mathbf{y}, \mathbf{z}$ are two fixed vectors in $\mathbf{R}^n$, then the mapping $T : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{y})\mathbf{y} + (\mathbf{x} \cdot \mathbf{z})\mathbf{z}$$

is linear.
B) No matter what the entries $a, b, c \in \mathbf{R}$ are, all the roots of the characteristic polynomial of the matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ are real.
C) If $A \in M_{2\times 5}(\mathbf{R})$, then for every $b \in \mathbf{R}^2$ the general solution of the equation $Ax = b$ contains three arbitrary constants.

D) If A is an $n \times n$ matrix and $\det(A^2 - 49I) = 0$, then there exists a nonzero vector x such that $Ax = 7x$.

E) If $A, B \in M_{n\times n}(\mathbf{R})$, and $B = Q^{-1}AQ$ for some invertible $n \times n$ matrix Q, then $\det(B) = \det(A)$.

F) If $A \in M_{6\times 9}(\mathbf{R})$ then $\dim \mathrm{Col}(A)$ can equal 8.

G) If $A \in M_{6\times 9}(\mathbf{R})$ then $\dim \mathrm{Nul}(A)$ is at most 6.

H) If $A \in M_{6\times 9}(\mathbf{R})$ and $\dim \mathrm{Nul}(A) = 4$, then $\dim \mathrm{Col}(A) = 5$.

I) If U is a 7×3 matrix whose columns $\{u_1, u_2, u_3\}$ form an orthonormal set in $\mathbf{R}^7$, then u_1, u_2, u_3 are eigenvectors of the 7×7 matrix UU^t.

J) If A is a 3×3 matrix with eigenvalues $\lambda = 2, 5, -7$ then Cramer's Rule can be used to solve $Ax = b$ for all $bin\mathbf{R}^3$.

VIII. (*more challenging*)

A) Show that there exists a basis $\{p_1(x), p_2(x), p_3(x)\}$ for $V = P_2(\mathbf{R})$ satisfying

$$\begin{array}{lll} p_1(0) = 1, & p_1(1) = 0, & p_1(2) = 0 \\ p_2(0) = 0, & p_2(1) = 1, & p_2(2) = 0 \\ p_3(0) = 0, & p_3(1) = 0, & p_3(2) = 1 \end{array}$$

Be sure to say how you know your polynomials form a basis for V.

B) Suppose you knew that the polynomial $q(x)$ satisfied $q(0) = 8$, $q(1) = -3$ and $q(2) = 4$. What would the expansion of $q(x)$ in terms of the basis $\{p_1(x), p_2(x), p_3(x)\}$ be?

C) Generalize what you did in part A to show that given any three distinct real numbers a, b, c, there exists a basis $\{p_1(x), p_2(x), p_3(x)\}$ of $V = P_2(\mathbf{R})$ satisfying

$$\begin{array}{lll} p_1(a) = 1, & p_1(b) = 0, & p_1(c) = 0 \\ p_2(a) = 0, & p_2(b) = 1, & p_2(c) = 0 \\ p_3(a) = 0, & p_3(b) = 0, & p_3(c) = 1 \end{array}$$

IX. (*more challenging*)

A) Show that if $\mathbf{x}$ is an eigenvector of a matrix A with eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of A^2.

B) Suppose that A is symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$, and corresponding eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ forming an orthonormal basis of $\mathbf{R}^n$. Let $P_1, \ldots, P_n$ be the matrices of the orthogonal projections onto the lines spanned by the eigenvectors of A. Explain why

$$A^2 = \lambda_1^2 P_1 + \cdots + \lambda_n^2 P_n,$$

using the Spectral Theorem.

C) Verify that the formula in part B is true in the case of the 2×2 symmetric matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$.