Mathematics 244 – Linear Algebra
Discussion 2 – "Compatible" Mappings
February 18, 2004

Background

Today we will begin to consider the main topic of the second unit of the course, namely mappings $T : V \to W$ where V and W are both vector spaces. It will turn out that the most interesting mappings here are ones that are "compatible" in a suitable sense with the vector space structures (that is the vector sum and scalar multiplication operations) in V and W. Part of the goal of today's discussion is to identify what this "compatibility" should mean.

Discussion Questions

A. Let $V = W = \mathbf{R}^2$, and consider the mapping $R_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ that takes each vector $\mathbf{x}$ to the vector $R_\theta(\mathbf{x})$ obtained by rotating by an angle of θ counterclockwise about the origin.

1) For instance, with $\theta = \pi/2$, explain why $R_{\pi/2}((1,0)) = (0,1)$ and $R_\theta((0,1)) = (-1,0)$. What is $R_{\pi/2}((-1/2, \sqrt{3}/2))$?
2) Show that for a general vector (x,y), $R_{\pi/2}((x,y)) = (-y,x)$.
3) Suppose you apply $R_{\pi/2}$ to a linear combination of two vectors and compute

$$R_{\pi/2}(c(x_1,y_1) + (x_2,y_2)) = R_{\pi/2}((cx_1 + x_2, cy_1 + y_2)).$$

How does the result relate to $R_{\pi/2}((x_1,y_1))$ and $R_{\pi/2}((x_2,y_2))$?
4) More generally, the rotation mapping R_θ is defined by

$$R_\theta((x,y)) = (\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y)$$

Suppose you apply R_θ to a linear combination of two vectors and compute

$$R_\theta(c(x_1,y_1) + (x_2,y_2)) = R_\theta((cx_1 + x_2, cy_1 + y_2)).$$

How does the result relate to $R_\theta((x_1,y_1))$ and $R_\theta((x_2,y_2))$?

B. Let $V = \mathbf{R}^3$, $W = \mathbf{R}^2$ and consider the mapping $L : V \to W$ defined by

$$L((x_1,x_2,x_3)) = (x_1 + 2x_2, 3x_1 + 4x_3).$$

1) What are $L((1,0,0)), L((0,1,0))$ and $L((0,0,1))$?
2) Suppose you apply L to a linear combination of two vectors and compute

$$L((cx_1 + y_1, cx_2 + y_2, cx_3 + y_3)).$$

How does the result relate to $L((x_1,x_2,x_3))$ and $L((y_1,y_2,y_3))$?

C. Now let $V = C^1(\mathbf{R})$ (the space of all continuously differentiable functions – $f \in V$ means f' exists and is continuous on $\mathbf{R}$). Let $W = C(\mathbf{R})$. Consider the mapping $D : V \to W$ defined by $D(f) = f'$ (that is D takes a function $f \in V$ to its derivative function $f' \in W$).

1) What is $D(5e^{7x} + \cos(3x))$? How does $D(5e^{7x} + \cos(3x))$ relate to $D(e^{7x})$ and $D(\cos(3x))$?
2) For general $f, g \in V$, how does $D(cf + g)$ relate to $D(f)$ and $D(g)$? Explain how this is similar to what you saw in A parts 3 and 4.

D. Let $V = C(\mathbf{R})$ and $W = \mathbf{R}$. Define $I : V \to W$ by $I(f) = \int_0^1 f(x)\, dx$ (that is I takes the function f to its definite integral over the interval $[0, 1]$).

1) What is $I(5e^{7x}+\cos(3x))$? How does $I(5e^{7x}+\cos(x^2))$ relate to $I(e^{7x})$ and $I(\cos(3x))$?
2) For general $f, g \in V$, how does $I(cf + g)$ relate to $I(f)$ and $I(g)$? Explain how this is similar to what you saw in A parts 3 and 4, and B part 2.

E. Thinking about A part 4, and parts 2 of B, C and D, what is the corresponding general definition of a property of $T : V \to W$? Explain why this property might appropriately be called a sort of "compatibility" with the vector space structures in V and W.

Assignment

Group write-ups due Monday, February 23.