Mathematics 244 – Linear Algebra
Discussion 4 – Inner Products
April 21, 2004

*Background*

We have already used the dot product of vectors in $\mathbf{R}^n$ on a number of occasions in this course. Recall that if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, then $\mathbf{x} \cdot \mathbf{y}$ is the scalar:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Today we want to look at some properties of this operation and its generalizations.

*Discussion Questions*

A) At the start of multivariable calculus, you saw essentially that all of the *geometry* of $\mathbf{R}^n$ "comes from" this dot product operation. How do you use the dot product to measure the *distance* between two points $P_1$ and $P_2$ in $\mathbf{R}^n$? How do you use the dot product to measure the *angle* between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$? How do you tell when two vectors are *orthogonal* (perpendicular)?

B) We want to take a slightly different "angle" in discussing the dot product now. Namely, we want to think of the dot product as a *mapping* from one space to another. What is the domain of the dot product mapping? What is the target space of the dot product mapping?

C) We also want to introduce an alternative notation for the dot product that is used in Chapter 4 of the text. We will write

$$\mathbf{x} \cdot \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Show using this notation that the dot product has the following properties:

1) ("bilinearity") For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n$, and all $c \in \mathbf{R}$, $\langle c\mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = c\langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ and $\langle \mathbf{x}, c\mathbf{y} + \mathbf{z} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$. Also, why is this property called "bilinearity"? How does it relate to the property of linearity for a mapping from one vector space to another?
2) ("symmetry") For all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
3) ("positive-definiteness") For all $\mathbf{x} \in \mathbf{R}^n$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equals 0 if and only if $\mathbf{x}$ is the zero vector.

D) A set of vectors $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is said to be *orthogonal* if $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0$ for all pairs $i \neq j$. $S$ is said to be *orthonormal* if $S$ is orthogonal, and in addition $\|x_i\| = \sqrt{\langle \mathbf{x}_i, \mathbf{x}_i \rangle} = 1$ for all $i$.

1) Show that an orthogonal set of nonzero vectors is linearly independent.
2) Show that if $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is any orthonormal basis for $\mathbf{R}^n$, then the scalars in the linear combination expressing an arbitrary vector $\mathbf{x}$ in terms of $\beta$:

$$\mathbf{x} = a_1 \mathbf{x}_1 + \cdots + a_n \mathbf{x}_n$$

are given by $a_i = \langle \mathbf{x}, \mathbf{x}_i \rangle$ for all $i$.

E) The three properties in question C form the definition of a more general concept called an *inner product* on a vector space $V$. Any mapping $\langle\ ,\ \rangle : V \times V \to \mathbf{R}$ satisfying them is called an inner product on $V$.

1) Show that the mapping $\langle\ ,\ \rangle : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(matrix product) is an inner product on $\mathbf{R}^2$ (different from the dot product!)

2) (An inner product from calculus – if you're taking the ODE/Applied Math sequence next year, you'll see this example again!) Let $V$ be the vector space of all continuous functions on the unit interval $[0, 1]$. Show that

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\ dx$$

defines an inner product on $V$. (Note: the hardest part here is to show that if $f$ is not the zero function, then $\int_0^1 (f(x))^2\ dx > 0$. You'll need to think about the definition of the definite integral and the properties of continuous functions here.)

3) Let $V = M_{2\times 2}(\mathbf{R})$ be the vector space of $2 \times 2$ matrices. Show that

$$\langle A, B \rangle = Tr(A^t B)$$

(that's the trace of the matrix product of $A$ transpose and $B$) defines an inner product.

*Assignment*

Group write-ups due in class on Monday, April 25.