Mathematics 244 – Linear Algebra
Discussion 3 – Linear Equations, revisited
March 17, 2004

Background

Today, we want to think through some of the connections between our work with systems of linear equations from Chapter 1 and the language of linear mappings, kernels, images, etc. from Chapter 2. To begin, recall from the end of class on Monday, that we have seen:

- If $T : \mathbf{R}^n \to \mathbf{R}^n$ is the linear mapping whose matrix with respect to the standard bases is

$$[T] = A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

  $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ is a vector in $\mathbf{R}^n$, and $\mathbf{b} = (b_1, b_2, \ldots, b_m)^t$ is a vector in $\mathbf{R}^m$, then the equation $T(\mathbf{x}) = \mathbf{b}$ is equivalent to the system of linear equations:

$$(1) \qquad \begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}.$$

- The system of equations (1) can be written in the shorthand form $A\mathbf{x} = \mathbf{b}$.
- Hence, $\mathbf{x} \in \text{Ker}(T)$ if and only if $A\mathbf{x} = \mathbf{0}$ – that is, if and only if $\mathbf{x}$ is a solution of the system (1) with $b_1 = \cdots = b_m = 0$.
- $\mathbf{b}$ is in $\text{Im}(T)$ if and only if the system (1) for that $\mathbf{b}$ *has at least one solution.*

Discussion Questions

A) Consider the echelon form system of equations $A\mathbf{x} = \mathbf{b}$ defined by

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 \end{pmatrix}$$

and $\mathbf{b} = (3, 7)^t$ (i.e. the column vector with these components).

1) Find parametrization of the set of solutions of this system.
2) Show that your set of solutions is the same as the set of vectors in $(3, 0, 7, 0, 0) + \text{Ker}(T)$ (the sum of a fixed vector and a subspace – this notation is the same as Exercise 5 from §1.2.)
3) Verify that the Dimension Theorem holds here.

4) Explain why this says that *the general solution* of $A\mathbf{x} = \mathbf{b}$ is any *particular solution*, plus the general solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

B) The pattern from part A is true in general. Suppose that $T : V \to W$ is any linear mapping, and assume that $\mathbf{b} \in \mathrm{Im}(T)$ so there is some $\mathbf{x}_0$ such that $T(\mathbf{x}_0) = \mathbf{b}$. Then show that the set S of all solutions $\mathbf{x}$ of the equation $T(\mathbf{x}) = \mathbf{b}$ is the set

$$S = \mathbf{x}_0 + \mathrm{Ker}(T)$$

(Hint: Show $S \subseteq \mathbf{x}_0 + \mathrm{Ker}(T)$ and $\mathbf{x}_0 + \mathrm{Ker}(T) \subseteq S$.)

C) The result from parts A and B also holds in situations like the following. Suppose we want to find all solution functions $y = y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} - e^x = 0 \tag{2}$$

Let $T : C^2(\mathbf{R}) \to C(\mathbf{R})$ be the mapping $T(f) = \frac{d^2 f}{dx^2}$. What is the kernel of T? Find the general solution of (2). Does the Dimension Theorem apply in this example? Why or why not?

D) The matrix-vector form $A\mathbf{x} = \mathbf{b}$ of the system (1) can also be used to formulate a "shortcut" method for the process of reducing a system of equations to echelon form and identifying the set of solutions. The *augmented matrix* of (1) is the $m \times (n+1)$ matrix

$$(A \mid \mathbf{b}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \mid b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \mid b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \mid b_m \end{pmatrix}$$

(just "tack" $\mathbf{b}$ "onto A" as an additional column). All of the arithmetic of reducing to the echelon form system can be done directly on this matrix (i.e. without explicitly writing the variables x_j for $j = 1, \ldots, n$ at each step). To see how this works, find the augmented matrix of the following system

$$\begin{array}{rcrcrcl} 3x_1 & - & 2x_2 & + & x_3 & = & 1 \\ x_1 & - & x_2 & + & 4x_3 & = & 0 \end{array}$$

carry out the operations to reduce to echelon forms directly on the rows of the matrix, and read off a parametrization of the set of solutions. *Also,* explain how your answer relates to the Dimension Theorem.

Assignment

Group writeups due on Monday, March 22.