Mathematics 244 – Linear Algebra
Answers – Final Exam
May 11, 2004

I.

A) Let $p, q \in W$. Then $p'(1) = q'(1) = 0$, $p(0) = 3p(2)$, and $q(0) = 3q(2)$. So if c is any constant, then $(cp+q)'(1) = cp'(1) + q'(1) = c \cdot 0 + 0$. Also,

$$(cp+q)(0) = cp(0) + q(0) = c \cdot 3p(2) + 3q(2) = 3(cp(2) + q(2)) = 3(cp+q)(2).$$

Since W is closed under sums and scalar multiples, W is a subspace of $P_4(\mathbf{R})$.

B) Let $p = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ be a general element of $P_4(\mathbf{R})$. Then $p \in W$ if and only if

$$\begin{aligned} p'(1) &= a_1 + 2a_2 + 3a_3 + 4a_4 = 0 \\ p(0) &= a_0 = 3p(2) = 3a_0 + 6a_1 + 12a_2 + 24a_3 + 48a_4 \end{aligned}$$

This says that the coefficients of p must satisfy

$$\begin{array}{ccccccccccc} & & a_1 & + & 2a_2 & + & 3a_3 & + & 4a_4 & = & 0 \\ 2a_0 & + & 6a_1 & + & 12a_2 & + & 24a_3 & + & 48a_4 & = & 0 \end{array}$$

Putting the system into echelon form, we get

$$\begin{array}{ccccccccccc} a_0 & & & & & + & 3a_3 & + & 12a_4 & = & 0 \\ & & a_1 & + & 2a_2 & + & 3a_3 & + & 4a_4 & = & 0 \end{array}$$

The free variables are a_2, a_3, a_4, so by the usual process we get

$$W = \mathrm{Span}\{x^2 - 2x, x^3 - 3x - 3, x^4 - 4x - 12\}$$

(the first comes by setting $a_2 = 1, a_3 = 0, a_4 = 0$, etc.)

C) This set is linearly independent, since if $c_1(x^2-2x)+c_2(x^3-3x-3)+c_3(x^4-4x-12) = 0$, then equating coefficients of x^2 shows $c_1 = 0$, equating coefficients of x^3 shows $c_2 = 0$, and equating coefficients of x^4 shows $c_3 = 0$.

II.

See class notes. The proof of (1.6.8) in the text follows almost the same reasoning as this proof.

III. A)

$$[T]_{std}^{std} = \begin{pmatrix} 4 & -3 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

since $T(e_1) = (4,0,1) = 4e_1 + 0e_2 + 1e_3$, etc.

B) We have

$$[T]_{\alpha'}^{\beta'} = [I_{\mathbf{R}^3}]_{std}^{\beta'}[T]_{std}^{std}[I_{\mathbf{R}^4}]_{\alpha'}^{std}$$

From the given bases α', β',

$$[I_{\mathbf{R}^4}]_{\alpha'}^{std} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 3 & -2 \end{pmatrix}$$

By the Gauss-Jordan procedure,

$$[I_{\mathbf{R}^3}]_{std}^{\beta'} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

So,

$$[T]_{\alpha'}^{\beta'} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & -3 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 3 & 0 & -7 \\ -6 & -2 & -3 & 5 \\ -7 & -1 & -1 & -4 \end{pmatrix}$$

IV. The corresponding echelon form system is

$$\left(\begin{array}{cccc|c} 1 & 0 & 6/7 & -1/7 & -2/7 \\ 0 & 1 & 3/7 & 3/7 & 6/7 \\ 0 & 0 & 0 & 0 & a-10 \end{array}\right)$$

This means the system is consistent if and only if $a = 10$. If so the solutions of the system are all vectors

$$\left(\frac{-2}{7}, \frac{6}{7}, 0, 0\right) + s\left(\frac{-6}{7}, \frac{-3}{7}, 1, 0\right) + t\left(\frac{1}{7}, \frac{-3}{7}, 0, 1\right)$$

where s, t are arbitrary real numbers.

V. The base case is when $A = (a_{11})$. By definition, $\det(A) = a_{11}$ is the product of the diagonal entries in this case. Now, assume that the result is true for all $k \times k$ upper-triangular matrices and consider an upper-triangular A which is $(k+1)\times(k+1)$. expanding along the first column, we see $\det(A) = a_{11}det(A_{11})+0$, since all the other entries in column

1 are zeroes. The (1,1)-minor A_{11} is a $k \times k$ upper-triangular matrix with diagonal entries $a_{22}, \ldots, a_{k+1,k+1}$, so the induction hypothesis shows that

$$\det(A) = a_{11} \det(A_{11}) = a_{11} a_{22} \cdots a_{k+1,k+1}$$

which establishes the induction step. Hence the result is true for all upper-triangular matrices.

VI. A) Expanding $\det(A - \lambda I)$ along row 3, we have

$$\det \begin{pmatrix} 2-\lambda & 9 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (5-\lambda) \det \begin{pmatrix} 2-\lambda & 9 \\ 1 & 2-\lambda \end{pmatrix} = -(5-\lambda)^2(\lambda+1)$$

so the eigenvalues are $\lambda = 5, 5, -1$.

B) We have

$$\begin{aligned} E_5 &= \operatorname{Ker} \begin{pmatrix} -3 & 9 & 0 \\ 1 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \operatorname{Ker} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \operatorname{Span}\{(3,1,0)\} \end{aligned}$$

Similarly,

$$\begin{aligned} E_{-1} &= \operatorname{Ker} \begin{pmatrix} 3 & 9 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \operatorname{Ker} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \operatorname{Span}\{(-3,1,0)\} \end{aligned}$$

C) The multiplicity of $\lambda = 5$ as a root of the characteristic polynomial is 2, but $\dim E_5 = 1$. Hence there is no way to get a linearly independent set of 3 eigenvectors of this A. Hence A is not diagonalizable.

VII. A) True: Because of the bilinearity of the dot product,

$$\begin{aligned} T(cx + w) &= \langle cx + w, y\rangle y + \langle cx + w, z\rangle z \\ &= c\langle x, y\rangle y + \langle w, y\rangle y + c\langle x, z\rangle z + \langle w, z\rangle z \\ &= c(\langle x, y\rangle y + \langle x, z\rangle z) + \langle w, y\rangle y + \langle w, z\rangle z \\ &= cT(x) + T(w) \end{aligned}$$

B) True, since the matrix is symmetric (see (4.5.6) in the text). (Many people worked *much, much too hard* on this one!)

C) False. If T is not surjective, then $\dim \mathrm{Im}(T) < 2$ and the Dimension Theorem says $\dim \mathrm{Ker}(T) > 3$. In this case, then

$$\begin{cases} y \notin \mathrm{Im}(T) \Rightarrow & \text{no solutions at all} \\ y \in \mathrm{Im}(T) \Rightarrow & \text{solutions contain more than 3 arbitrary consts.} \end{cases}$$

For example, say $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_1)$ is a mapping $T : \mathbf{R}^5 \to \mathbf{R}^2$. The image consists of all vectors whose components are equal. If $(1,1)$ is such a vector then $T(x_1, x_2, x_3, x_4, x_5) = (1,1)$ has general solution

$$(x_1, x_2, x_3, x_4, x_5) = (1,0,0,0,0)+s(0,1,0,0,0)+t(0,0,1,0,0)+u(0,0,0,1,0)+v(0,0,0,0,1)$$

where s, t, u, v are arbitrary. For other vectors like $(2,3)$, $T(x_1, x_2, x_3, x_4, x_5) = (2,3)$ has no solutions at all.

VIII. A) The main point of this problem is to derive a way to write down the desired polynomials. The most economical way to do this is to note that p_1 must have roots at $x = 1, 2$, p_2 must have roots at $x = 0, 2$, etc. This says

$$p_1(x) = c_1(x-1)(x-2),\ p_2(x) = c_2 x(x-2),\ p_3(x) = c_3 x(x-1)$$

for some constants c_i. Then the values of c_i can be determined by substituting the other x-value and using the given information

$$p_1(0) = 1 \Rightarrow c_1(0-1)(0-2) = 1 \Rightarrow c_1 = 1/2$$

and similarly, $c_2 = -1$, $c_3 = 1/2$. The polynomials are

$$\begin{aligned} p_1(x) &= \frac{1}{2}x^2 - \frac{3}{2}x + 1 \\ p_2(x) &= -x^2 + 2x \\ p_3(x) &= \frac{1}{2}x^2 - \frac{1}{2}x \end{aligned}$$

To check that these are linearly independent, we consider a linear combination,

$$0 = a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x)$$

Substitute $x = 0$, then we get $0 = a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0$. Hence $a_1 = 0$. Similarly, substituting $x = 1$ shows $a_2 = 0$, $x = 2$ shows $a_3 = 0$. Since $\dim P_2(\mathbf{R}) = 3$, and this is an independent subset with 3 elements, it is a basis.

B) Write

$$q(x) = a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x)$$

As above, substitute $x = 0, 1, 2$ one after another. $x = 0$ gives

$$8 = q(0) = a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0$$

Hence $a_1 = 8$. Similarly, $a_2 = -3$, $a_3 = 4$.

C) Following the idea of proof given in part A above,

$$p_1(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)}$$
$$p_2(x) = \frac{(x-a)(x-c)}{(b-a)(b-c)}$$
$$p_1(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)}$$

This is a special case of a general result called the *Lagrange Interpolation Theorem.*

VIII'. A) If $Ax = \lambda x$, then

$$A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$$

Hence x is also an eigenvector of A^2 with eigenvalue λ^2.

B) Since A is symmetric, so is A^2 (reason: $(A^2)^t = (AA)^t = A^t A^t = A^2$). Apply the Spectral Theorem to the symmetric matrix A^2. Part A says that the eigenspaces and the projections for A^2 are the same as those for A (the eigenspaces are equal). But the eigenvalues for A^2 are the λ^2. Hence the spectral decomposition for A^2 is

$$A^2 = \lambda_1^2 P_1 + \cdots + \lambda_k^2 P_k$$

(Alternate method: From the spectral decomposition for A,

$$A = \lambda_1 P_1 + \cdots + \lambda_k P_k$$

So

$$A^2 = (\lambda_1 P_1 + \cdots + \lambda_k P_k)(\lambda_1 P_1 + \cdots + \lambda_k P_k)$$

Matrix product is distributive over matrix sum, so this can be written as

$$A^2 = \lambda_1^2 P_1^2 + \cdots + \lambda_k^2 P_k^2 + \sum_{i \neq j} \lambda_i \lambda_j P_i P_j$$

But $P_i^2 = P_i$ and if $i \neq j$, then $P_i P_j = 0$. This follows because as we know, eigenvectors for distinct eigenvalues of a symmetric matrix are orthogonal, hence $\mathrm{Im}(P_j) = E_{\lambda_j} \subset (E_{\lambda_i})^\perp = \mathrm{Ker}(P_i)$. So when you apply $P_i P_j$ to any vector you will get zero.)

C) The eigenvalues of A are $\lambda = 3, 5$. Correxponding bases for the eigenspaces are $\{(1,1)\}$ for E_5 and $\{(-1,1)\}$ for E_3.

$$P_{E_5} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix})^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Similarly,

$$P_{E_3} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} (\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix})^{-1} \begin{pmatrix} -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

So

$$A^2 = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix} = 25 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 9 \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$