I.

A) Let \( p, q \in W \). Then \( p'(1) = q'(1) = 0 \), \( p(0) = 3p(2) \), and \( q(0) = 3q(2) \). So if \( c \) is any constant, then \( (cp + q)'(1) = cp'(1) + q'(1) = c \cdot 0 + 0 \). Also,

\[
(cp + q)(0) = cp(0) + q(0) = c \cdot 3p(2) + 3q(2) = 3(cp(2) + q(2)) = 3(cp + q)(2).
\]

Since \( W \) is closed under sums and scalar multiples, \( W \) is a subspace of \( P_4(\mathbb{R}) \).

B) Let \( p = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \) be a general element of \( P_4(\mathbb{R}) \). Then \( p \in W \) if and only if

\[
\begin{align*}
p'(1) &= a_1 + 2a_2 + 3a_3 + 4a_4 = 0 \\
p(0) &= a_0 = 3p(2) = 3a_0 + 6a_1 + 12a_2 + 24a_3 + 48a_4
\end{align*}
\]

This says that the coefficients of \( p \) must satisfy

\[
\begin{align*}
a_0 &+ 2a_2 &+ 3a_3 &+ 4a_4 &= 0 \\
3a_0 &+ 2a_2 &+ 3a_3 &+ 4a_4 &= 0
\end{align*}
\]

Putting the system into echelon form, we get

\[
\begin{align*}
a_0 &+ 3a_3 &+ 12a_4 &= 0 \\
a_1 &+ 2a_2 &+ 3a_3 &+ 4a_4 &= 0
\end{align*}
\]

The free variables are \( a_2, a_3, a_4 \), so by the usual process we get

\[
W = \text{Span}\{x^2 - 2x, x^3 - 3x - 3, x^4 - 4x - 12\}
\]

(the first comes by setting \( a_2 = 1, a_3 = 0, a_4 = 0 \), etc.)

C) This set is linearly independent, since if \( c_1(x^2 - 2x) + c_2(x^3 - 3x - 3) + c_3(x^4 - 4x - 12) = 0 \), then equating coefficients of \( x^2 \) shows \( c_1 = 0 \), equating coefficients of \( x^3 \) shows \( c_2 = 0 \), and equating coefficients of \( x^4 \) shows \( c_3 = 0 \).

II.

See class notes. The proof of (1.6.8) in the text follows almost the same reasoning as this proof.

III. A)

\[
[T]_{std}^{std} = \begin{pmatrix} 4 & -3 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}
\]
since \( T(c_1) = (4, 0, 1) = 4c_1 + 0c_2 + 1c_3, \) etc.

B) We have

\[
[T]^\beta_{\alpha'} = [I_{\mathbb{R}^3}]^\beta_{\alpha'}^\text{std} [T]^\text{std} [I_{\mathbb{R}^4}]^\text{std}_{\alpha'}
\]

From the given bases \( \alpha', \beta' \),

\[
[I_{\mathbb{R}^4}]^\text{std}_{\alpha'} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & 3 & -2 \\
\end{pmatrix}
\]

By the Gauss-Jordan procedure,

\[
[I_{\mathbb{R}^3}]^\beta_{\alpha'}^\text{std} = \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & 1 & 1 \\
-1 & -1 & 0 \\
-1 & 0 & 0 \\
\end{pmatrix}
\]

So,

\[
[T]^\beta_{\alpha'} = \begin{pmatrix}
1 & 1 & 1 \\
-1 & -1 & 0 \\
-1 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
4 & -3 & 2 & -1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
\end{pmatrix} \begin{pmatrix}
1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & 3 & -2 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
7 & 3 & 0 & -7 \\
-6 & -2 & -3 & 5 \\
-7 & -1 & -1 & -4 \\
\end{pmatrix}
\]

IV. The corresponding echelon form system is

\[
\begin{pmatrix}
1 & 0 & 6/7 & -1/7 & -2/7 \\
0 & 1 & 3/7 & 3/7 & 6/7 \\
0 & 0 & 0 & 0 & a - 10 \\
\end{pmatrix}
\]

This means the system is consistent if and only if \( a = 10 \). If so the solutions of the system are all vectors

\[
\left( -\frac{2}{7}, \frac{6}{7}, 0, 0 \right) + s \left( -\frac{6}{7}, -\frac{3}{7}, 1, 0 \right) + t \left( \frac{1}{7}, -\frac{3}{7}, 0, 1 \right)
\]

where \( s, t \) are arbitrary real numbers.

V. The base case is when \( A = (a_{11}) \). By definition, \( \det(A) = a_{11} \) is the product of the diagonal entries in this case. Now, assume that the result is true for all \( k \times k \) upper-triangular matrices and consider an upper-triangular \( A \) which is \((k+1) \times (k+1)\). expanding along the first column, we see \( \det(A) = a_{11} \det(A_{11}) + 0 \), since all the other entries in column
1 are zeroes. The \((1,1)\)-minor \(A_{11}\) is a \(k \times k\) upper-triangular matrix with diagonal entries \(a_{22}, \ldots, a_{k+1,k+1}\), so the induction hypothesis shows that
\[
\det(A) = a_{11} \det(A_{11}) = a_{11} a_{22} \cdots a_{k+1,k+1}
\]
which establishes the induction step. Hence the result is true for all upper-triangular matrices.

VI. A) Expanding \(\det(A - \lambda I)\) along row 3, we have
\[
\det \begin{pmatrix} 2 - \lambda & 9 & 0 \\ 1 & 2 - \lambda & -1 \\ 0 & 0 & 5 - \lambda \end{pmatrix} = (5 - \lambda) \det \begin{pmatrix} 2 - \lambda & 9 \\ 1 & 2 - \lambda \end{pmatrix} = -(5 - \lambda)^2(\lambda + 1)
\]
so the eigenvalues are \(\lambda = 5, 5, -1\).

B) We have
\[
E_5 = \text{Ker} \begin{pmatrix} -3 & 9 & 0 \\ 1 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Ker} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\{(3, 1, 0)\}
\]
Similarly,
\[
E_{-1} = \text{Ker} \begin{pmatrix} 3 & 9 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Ker} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\{(-3, 1, 0)\}
\]
C) The multiplicity of \(\lambda = 5\) as a root of the characteristic polynomial is 2, but \(\dim E_5 = 1\). Hence there is no way to get a linearly independent set of 3 eigenvectors of this \(A\). Hence \(A\) is not diagonalizable.

VII. A) True: Because of the bilinearity of the dot product,
\[
T(cx + w) = \langle cx + w, y \rangle y + \langle cx + w, z \rangle z \\
= c \langle x, y \rangle y + \langle w, y \rangle y + c \langle x, z \rangle z + \langle w, z \rangle z \\
= c(\langle x, y \rangle y + \langle x, z \rangle z) + \langle w, y \rangle y + \langle w, z \rangle z \\
= cT(x) + T(w)
\]
B) True, since the matrix is symmetric (see (4.5.6) in the text). (Many people worked much, much too hard on this one!)

C) False. If $T$ is not surjective, then $\dim \text{Im}(T) < 2$ and the Dimension Theorem says $\dim \text{Ker}(T) > 3$. In this case, then

$$\left\{ \begin{array}{l} y \notin \text{Im}(T) \Rightarrow \text{no solutions at all} \\ y \in \text{Im}(T) \Rightarrow \text{solutions contain more than 3 arbitrary consts.} \end{array} \right.$$  

For example, say $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_1)$ is a mapping $T : \mathbb{R}^5 \to \mathbb{R}^2$. The image consists of all vectors whose components are equal. If $(1, 1)$ is such a vector then $T(x_1, x_2, x_3, x_4, x_5) = (1, 1)$ has general solution

$$(x_1, x_2, x_3, x_4, x_5) = (1, 0, 0, 0, 0) + s(0, 1, 0, 0, 0) + t(0, 0, 1, 0, 0) + u(0, 0, 0, 1, 0) + v(0, 0, 0, 0, 1)$$

where $s, t, u, v$ are arbitrary. For other vectors like $(2, 3), T(x_1, x_2, x_3, x_4, x_5) = (2, 3)$ has no solutions at all.

VIII. A) The main point of this problem is to derive a way to write down the desired polynomials. The most economical way to do this is to note that $p_1$ must have roots at $x = 1, 2, p_2$ must have roots at $x = 0, 2$, etc. This says

$$p_1(x) = c_1(x - 1)(x - 2), \quad p_2(x) = c_2x(x - 2), \quad p_3(x) = c_3x(x - 1)$$

for some constants $c_i$. Then the values of $c_i$ can be determined by substituting the other $x$-value and using the given information

$$p_1(0) = 1 \Rightarrow c_1(0 - 1)(0 - 2) = 1 \Rightarrow c_1 = 1/2$$

and similarly, $c_2 = -1, \ c_3 = 1/2$. The polynomials are

$$p_1(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1 \quad \quad \quad p_2(x) = -x^2 + 2x \quad \quad \quad p_3(x) = \frac{1}{2}x^2 - \frac{1}{2}x$$

To check that these are linearly independent, we consider a linear combination,

$$0 = a_1p_1(x) + a_2p_2(x) + a_3p_3(x)$$

Substitute $x = 0$, then we get $0 = a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0$. Hence $a_1 = 0$. Similarly, substituting $x = 1$ shows $a_2 = 0, x = 2$ shows $a_3 = 0$. Since $\dim P_2(\mathbb{R}) = 3$, and this is an independent subset with 3 elements, it is a basis.

B) Write

$$q(x) = a_1p_1(x) + a_2p_2(x) + a_3p_3(x)$$
As above, substitute \( x = 0,1,2 \) one after another. \( x = 0 \) gives

\[
8 = q(0) = a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0
\]

Hence \( a_1 = 8 \). Similarly, \( a_2 = -3, \ a_3 = 4 \).

C) Following the idea of proof given in part A above,

\[
p_1(x) = \frac{(x - b)(x - c)}{(a - b)(a - c)}
\]

\[
p_2(x) = \frac{(x - a)(x - c)}{(b - a)(b - c)}
\]

\[
p_1(x) = \frac{(x - a)(x - b)}{(c - a)(c - b)}
\]

This is a special case of a general result called the \textit{Lagrange Interpolation Theorem}.

VIII. A) If \( Ax = \lambda x \), then

\[
A^2x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x
\]

Hence \( x \) is also an eigenvector of \( A^2 \) with eigenvalue \( \lambda^2 \).

B) Since \( A \) is symmetric, so is \( A^2 \) (reason: \( (A^2)^t = (AA)^t = A^tA^t = A^2 \)). Apply the Spectral Theorem to the symmetric matrix \( A^2 \). Part A says that the eigenspaces and the projections for \( A^2 \) are the same as those for \( A \) (the eigenspaces are equal). But the eigenvalues for \( A^2 \) are the \( \lambda^2 \). Hence the spectral decomposition for \( A^2 \) is

\[
A^2 = \lambda_1^2 P_1 + \cdots + \lambda_k^2 P_k
\]

(Alternate method: From the spectral decomposition for \( A \),

\[
A = \lambda_1 P_1 + \cdots + \lambda_k P_k
\]

So

\[
A^2 = (\lambda_1 P_1 + \cdots + \lambda_k P_k)(\lambda_1 P_1 + \cdots + \lambda_k P_k)
\]

Matrix product is distributive over matrix sum, so this can be written as

\[
A^2 = \lambda_1^2 P_1^2 + \cdots + \lambda_k^2 P_k^2 + \sum_{i \neq j} \lambda_i \lambda_j P_i P_j
\]

But \( P_i^2 = P_i \) and if \( i \neq j \), then \( P_i P_j = 0 \). This follows because as we know, eigenvectors for distinct eigenvalues of a symmetric matrix are orthogonal, hence \( \text{Im}(P_j) = E_{\lambda_j} \subset (E_{\lambda_i})^\perp = \text{Ker}(P_i) \). So when you apply \( P_i P_j \) to any vector you will get zero.)
C) The eigenvalues of $A$ are $\lambda = 3, 5$. Corresponding bases for the eigenspaces are $\{(1, 1)\}$ for $E_5$ and $\{(-1, 1)\}$ for $E_3$.

$$P_{E_5} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix})^{-1} (1 \\ 1) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Similarly,

$$P_{E_3} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} (\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix})^{-1} (-1 \\ 1) = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

So

$$A^2 = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix} = 25 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 9 \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$