A) If \( T : V \to W \) is linear, \( \dim(V) = 5, \dim(W) = 7 \), then \( T \) is surjective.

This is \textit{FALSE}. For instance, letting \( T \) be the zero mapping gives a counterexample since \( \text{Im}(T) = \{0_W\} \neq W \).

B) Every linear \( T : V \to W \), \( \dim(V) = 5, \dim(W) = 7 \), is \( T \) is injective.

This is \textit{FALSE}. For instance, letting \( T \) be the zero mapping gives a counterexample since \( \text{Ker}(T) = V \neq \{0_V\} \).

C) Some linear \( T : V \to W \) is linear, \( \dim(V) = 5, \dim(W) = 7 \), then \( T \) is injective.

This is \textit{TRUE}. For instance, let \( \beta = \{v_1, \ldots, v_5\} \) be a basis for \( V \), and let \( \gamma = \{w_1, \ldots, w_5\} \) be any linearly independent subset of \( W \). Then the unique linear \( T : V \to W \) with \( T(v_i) = w_i \), \( i = 1, \ldots, 5 \) is injective. \textit{Proof}: We show this by showing \( T \) satisfies \( \text{Ker}(T) = \{0_V\} \). Suppose \( T(v) = 0_W \). Then since \( \beta \) is a basis for \( V \),

\[ v = a_1v_1 + \cdots + a_5v_5 \]

for some \( a_i \in \mathbb{R} \). But then, since \( T \) is linear,

\[ 0_W = T(v) = a_1T(v_1) + \cdots + a_5T(v_5) \]

Since \( \gamma \) is linearly independent, this implies \( a_i = 0 \) for all \( i \), which says \( v = 0_V \). Hence \( \text{Ker}(T) = \{0_V\} \).

D) If \( \{v_1, \ldots, v_n\} \) in \( V \) is linearly dependent, and \( T : V \to W \) is linear then \( \{T(v_1), \ldots, T(v_n)\} \) is linearly dependent.

This is \textit{TRUE}. If \( \{v_1, \ldots, v_n\} \) is linearly dependent, then there exist scalars \( a_i \), not all equal to zero, such that

\[ a_1v_1 + \cdots + a_nv_n = 0_V \]

But then \( T \) linear shows

\[ T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n) = T(0_V) = 0_W \]

This implies that \( \{T(v_1), \ldots, T(v_n)\} \) is linearly dependent since some \( a_i \neq 0 \).

E) If \( T : \mathbb{R}^n \to \mathbb{R}^n \) is linear then \( \text{Ker}(T) \cap \text{Im}(T) = \{0\} \).
This is FALSE. A counterexample is given by the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix

$$[T] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(as in a problem from the last problem set). For this $T$, $\text{Ker}(T) = \text{Im}(T) = \text{Span}((1, 0))$, so $\text{Ker}(T) \cap \text{Im}(T) = \text{Span}((1, 0))$.

F) If $T : V \to V$ is linear and $\beta = \{v_i\}$ and $\beta' = \{cv_i\}$ for some scalar $c \neq 0$, then $[T]_{\beta}^{\beta'} = [T]_{\beta'}^{\beta}$.

This is TRUE. The scalars in the $j$th column of $[T]_{\beta}^{\beta'}$ are the $a_{ij}$ in

$$T(v_j) = a_{1j}v_1 + \cdots + a_{nj}v_n$$

By linearity and one of the “distributive laws” for scalar multiplication:

$$T(cv_j) = cT(v_j)$$

$$= c(a_{1j}v_1 + \cdots + a_{nj}v_n)$$

$$= a_{1j}(cv_1) + \cdots + a_{nj}(cv_n)$$

Hence the $j$th column of $[T]_{\beta}^{\beta'}$ is the same as the $j$th column of $[T]_{\beta}^{\beta}$ for all $j$, so the matrices are equal.

G) If $A$ is an $n \times n$ matrix and $c$ is a fixed real number then the set $W = \{v \in \mathbb{R}^n : Av = cv\}$ is a vector subspace of $\mathbb{R}^n$.

This is TRUE. Let $v, w \in W$ and let $d$ be an arbitrary scalar. Then by distributivity of matrix multiplication over matrix sums,

$$A(dv + w) = dAv + Aw$$

$$= dcv + cw$$

$$= c(dv + w)$$

Hence $dv + w \in W$, and $W$ is a vector subspace.

H) If $T = I_{\mathbb{R}^2}$ is the identity mapping, and $\beta, \beta'$ are two bases for $\mathbb{R}^2$, then

$$[T]_{\beta}^{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is FALSE. A counterexample is: Let $\beta = \{(3, 4), (1, -1)\}$, and let $\beta'$ be the standard basis. Then

$$[T]_{\beta}^{\beta'} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}$$
I) If $T : V \rightarrow W$ is an isomorphism, and $U$ is a vector subspace of $V$, then $\dim(U) = \dim(T(U))$.

This is TRUE. If $\{v_1, ..., v_n\}$ is a basis for $U$, then we know $\{T(v_1), ..., T(v_n)\}$ spans $T(U)$. Furthermore, since $T$ is an isomorphism, it is injective. Hence, if $c_1 T(v_1) + \cdots + c_n T(v_n) = 0$, then $T(c_1 v_1 + \cdots + c_n v_n) = 0$ which implies $c_1 v_1 + \cdots + c_n v_n = 0$, which implies $c_i = 0$ for $i = 1, ..., n$. Thus $\{T(v_1), ..., T(v_n)\}$ is linearly independent, hence a basis for $\text{Im}(U)$. Hence the dimensions of $U$ and $T(U)$ are equal.

J) If $v_1, ..., v_n$ are vectors in a vector space $V$, and $w_1, ..., w_n$ are vectors in a vector space $W$, then there always exists some linear $T : V \rightarrow W$ satisfying $T(v_i) = w_i$ for all $i = 1, ..., n$.

This is FALSE because we could consider a case where the $v_i$ are linearly dependent, but the $w_i$ are linearly independent. There is no linear mapping that takes a linearly dependent set to a linearly independent set (see question D above).

K) If $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear, and $\ker(T) + \text{im}(T) = \mathbb{R}^n$, then $\ker(T) \cap \text{im}(T) = \{0\}$.

This is TRUE. By the Dimension Theorem, $\dim \ker(T) + \dim \text{im}(T) = n$. The equality $\ker(T) + \text{im}(T) = \mathbb{R}^n$ is given. Hence if $\beta$ is a basis for $\ker(T)$ and $\gamma$ is a basis for $\text{im}(T)$, then $\beta \cup \gamma$ is a basis for $\mathbb{R}^n$, because it spans $\mathbb{R}^n$ and contains exactly $n$ vectors. If $\ker(T) \cap \text{im}(T)$ contained some nonzero vector, $v$, then there would be an equation $v = w$ for some $v \in \ker(T)$ and some $w \in \text{im}(T)$ but then from $v - w = 0$, expanding $v$ in terms of $\beta$ and $w$ in terms of $\gamma$, there would be a linear dependence on the vectors in $\beta \cup \gamma$. This is impossible, so $\ker(T) \cap \text{im}(T) = \{0\}$. 

(look at the columns).