Mathematics 244 – Linear Algebra
Solutions – Review True-False Questions for Exam 2
March 24, 2004

A) If $T : V \to W$ is linear, $\dim(V) = 5$, $\dim(W) = 7$, then T is surjective.

This is *FALSE*. For instance, letting T be the zero mapping gives a counterexample since $\mathrm{Im}(T) = \{\mathbf{0}_W\} \neq W$.

B) Every linear $T : V \to W$, $\dim(V) = 5$, $\dim(W) = 7$, is T is injective.

This is *FALSE*. For instance, letting T be the zero mapping gives a counterexample since $\mathrm{Ker}(T) = V \neq \{\mathbf{0}_V\}$.

C) Some linear $T : V \to W$ is linear, $\dim(V) = 5$, $\dim(W) = 7$, then T is injective.

This is *TRUE*. For instance, let $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ be a basis for V, and let $\gamma = \{\mathbf{w}_1, \ldots, \mathbf{w}_5\}$ be any linearly independent subset of W. Then the unique linear $T : V \to W$ with $T(\mathbf{v}_i) = \mathbf{w}_i$, $i = 1, \ldots, 5$ is injective. *Proof*: We show this by showing T satisfies $\mathrm{Ker}(T) = \{\mathbf{0}_V\}$. Suppose $T(\mathbf{v}) = \mathbf{0}_W$. Then since β is a basis for V,

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_5\mathbf{v}_5$$

for some $a_i \in \mathbf{R}$. But then, since T is linear,

$$\mathbf{0}_W = T(\mathbf{v}) = a_1T(\mathbf{v}_1) + \cdots + a_5T(\mathbf{v}_5)$$

Since γ is linearly independent, this implies $a_i = 0$ for all i, which says $\mathbf{v} = \mathbf{0}_V$. Hence $\mathrm{Ker}(T) = \{\mathbf{0}_V\}$.

D) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in V is linearly dependent, and $T : V \to W$ is linear then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly dependent.

This is *TRUE*. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent, then there exist scalars a_i, not all equal to zero, such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}_V$$

But then T linear shows

$$T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + \cdots + a_nT(\mathbf{v}_n) = T(\mathbf{0}_V) = \mathbf{0}_W$$

This implies that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly dependent since some $a_i \neq 0$.

E) If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear then $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \{\mathbf{0}\}$.

This is *FALSE*. A counterexample is given by the mapping $T : \mathbf{R}^2 \to \mathbf{R}^2$ with standard matrix

$$[T] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(as in a problem from the last problem set). For this T, $\mathrm{Ker}(T) = \mathrm{Im}(T) = \mathrm{Span}((1,0))$, so $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \mathrm{Span}((1,0))$.

F) If $T : V \to V$ is linear and $\beta = \{\mathbf{v}_i\}$ and $\beta' = \{c\mathbf{v}_i\}$ for some scalar $c \neq 0$, then $[T]_\beta^\beta = [T]_{\beta'}^{\beta'}$.

This is *TRUE*. The scalars in the jth column of $[T]_\beta^\beta$ are the a_{ij} in

$$T(\mathbf{v}_j) = a_{1j}\mathbf{v}_1 + \cdots + a_{nj}\mathbf{v}_n$$

By linearity and one of the "distributive laws" for scalar multiplication:

$$\begin{aligned} T(c\mathbf{v}_j) &= cT(\mathbf{v}_j) \\ &= c(a_{1j}\mathbf{v}_1 + \cdots + a_{nj}\mathbf{v}_n) \\ &= a_{1j}(c\mathbf{v}_1) + \cdots + a_{nj}(c\mathbf{v}_n) \end{aligned}$$

Hence the jth column of $[T]_{\beta'}^{\beta'}$ is the same as the jth column of $[T]_\beta^\beta$ for all j, so the matrices are equal.

G) If A is an $n \times n$ matrix and c is a fixed real number then the set $W = \{\mathbf{v} \in \mathbf{R}^n : A\mathbf{v} = c\mathbf{v}\}$ is a vector subspace of $\mathbf{R}^n$.

This is *TRUE*. Let $\mathbf{v}, \mathbf{w} \in W$ and let d be an arbitrary scalar. Then by distributivity of matrix multiplication over matrix sums,

$$\begin{aligned} A(d\mathbf{v} + \mathbf{w}) &= dA\mathbf{v} + A\mathbf{w} \\ &= dc\mathbf{v} + c\mathbf{w} \\ &= c(d\mathbf{v} + \mathbf{w}) \end{aligned}$$

Hence $d\mathbf{v} + \mathbf{w} \in W$, and W is a vector subspace.

H) If $T = I_{\mathbf{R}^2}$ is the identity mapping, and β, β' are two bases for $\mathbf{R}^2$, then

$$[T]_\beta^{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is *FALSE*. A counterexample is: Let $\beta = \{(3,4), (1,-1)\}$, and let β' be the standard basis. Then

$$[T]_\beta^{\beta'} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}$$

(look at the columns).

I) If $T : V \to W$ is an isomorphism, and U is a vector subspace of V, then $\dim(U) = \dim(T(U))$.

This is *TRUE*. If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis for U, then we know $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_n)\}$ spans $T(U)$. Furthermore, since T is an isomorphism, it is injective. Hence, if $c_1 T(\mathbf{v}_1) + \cdots + c_n T(\mathbf{v}_n) = \mathbf{0}$, then $T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}$ which implies $c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0}$, which implies $c_i = 0$ for $i = 1, ...n$. Thus $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_n)\}$ is linearly independent, hence a basis for $\mathrm{Im}(U)$. Hence the dimensions of U and $T(U)$ are equal.

J) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in a vector space V, and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are vectors in a vector space W, then there always exists some linear $T : V \to W$ satisfying $T(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 1, ..., n$.

This is *FALSE* because we could consider a case where the $\mathbf{v}_i$ are linearly dependent, but the $\mathbf{w}_i$ are linearly independent. There is no linear mapping that takes a linearly dependent set to a linearly independent set (see question D above).

K) If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear, and $\mathrm{Ker}(T) + \mathrm{Im}(T) = \mathbf{R}^n$, then $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \{\mathbf{0}\}$.

This is *TRUE*. By the Dimension Theorem, $\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = n$. The equality $\mathrm{Ker}(T) + Im(T) = \mathbf{R}^n$ is given. Hence if β is a basis for $\mathrm{Ker}(T)$ and γ is a basis for $\mathrm{Im}(T)$, then $\beta \cup \gamma$ is a basis for $\mathbf{R}^n$, because it spans $\mathbf{R}^n$ and contains exactly n vectors. If $\mathrm{Ker}(T) \cap \mathrm{Im}(T)$ contained some nonzero vector, $\mathbf{v}$, then there would be an equation $\mathbf{v} = \mathbf{w}$ for some $\mathbf{v} \in \mathrm{Ker}(T)$ and some $\mathbf{w} \in \mathrm{Im}(T)$ but then from $\mathbf{v} - \mathbf{w} = \mathbf{0}$, expanding $\mathbf{v}$ in terms of β and $\mathbf{w}$ in terms of γ, there would be a linear dependence on the vectors in $\beta \cup \gamma$. This is impossible, so $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \{\mathbf{0}\}$.