MATH 400 – Directed Readings in Module and Representation Theory
Midterm Problem Set – Due Friday, March 19 (no later than 5:00pm)

Directions and Groundrules: You may consult Dummit and Foote for definitions, background, etc. Do not consult other books, other faculty members, etc.

I. Let R be a commutative ring with $1 \neq 0$. Recall that an ideal $M \neq R$ in R is said to be *maximal* if whenever I is an ideal of R such that $M \subseteq I \subseteq R$, then either $I = M$ or $I = R$.

A) Show that there exist maximal ideals in R.
B) Show that if M is an ideal in R, then M is maximal if and only if R/M is a field.
C) Let $S = R \oplus \cdots \oplus R$ (n terms), as a module over R. If M is any ideal in R, show that $N = M \oplus \cdots \oplus M$ is an submodule in S and identify the quotient S/N up to isomorphism.
D) In this part and the next, $m, n \in \mathbf{N}$. Show that if F is a field, then $F^n \simeq F^m$ as vector spaces over F if and only if $n = m$.
E) Show that $R^n \simeq R^m$ (isomorphic as R-modules) if and only if $n = m$.

II. A *local ring* is a commutative ring with 1 which has a unique maximal ideal.

A) Show that $\mathbf{Z}/p^k\mathbf{Z}$ is a local ring for any prime p and $k \geq 1$, by identifying the unique maximal ideal.
B) Let F be a field and let $F[x]_{(x)}$ be the subring of the field of rational functions $F(x)$ consisting of "fractions" $f(x)/g(x)$ such that $g(x) \notin (x)$ (that is such that $g(0) \neq 0$). (You can think of $F[x]_{(x)}$ as the subring of the field $F(x)$ of rational functions for which evaluation at $x = 0$ makes sense.) Show that $F[x]_{(x)}$ is a local ring and identify the maximal ideal.
C) An important fact about modules over local rings is the following: Show that if M is a finitely generated module over a local ring R with unique maximal ideal I such that $IM = M$, then $M = \{0\}$.
D) Deduce that if M is a finitely generated module over the local ring R with unique maximal ideal I and N is a submodule such that $M = N + IM$, then $N = M$.
E) Deduce that if M is a finitely generated module over a local ring R with maximal ideal I and $x_1 + IM, \ldots, x_s + IM$ generate M/IM, then $x_1, \ldots, x_s$ generate M.

III. Let I be a nonempty index set and for each $i \in I$, let N_i be a submodule of the R-module M. Show that the following are equivalent:

(i) The submodule of M generated by the N_i is isomorphic to the direct sum $\oplus_{i\in I} N_i$.
(ii) If $i_1, \ldots, i_k$ is any finite subset of I, then $N_{i_1} \cap (N_{i_2} + \cdots + N_{i_k}) = \{0\}$.
(iii) If $i_1, \ldots, i_k$ is any finite subset of I, then $N_{i_1} + \cdots + N_{i_k}) \simeq N_{i_1} \oplus \cdots \oplus N_{i_k}$.
(iv) For any element x of the submodule generated by the N_i, there are unique $a_i \in N_i$, all but finitely many of which are zero, and such that $x = \sum_{i\in I} a_i$ (note this is a finite sum).

IV. Let R be a commutative ring with 1.

A) Prove that if M, N are free R-modules, then $M \otimes_R N$ is free.
B) Deduce that if M, N are projective, then $M \otimes_R N$ is projective.

C) Show that the polynomial ring $R[x]$ is a flat module over R.

V. Let A be an $n \times n$ matrix with entries in F that satisfies $A^2 = A$.

A) If $n = 5$, find all possible rational canonical forms of A.

B) For all n, show that $F^n = \ker(A) \oplus \operatorname{im}(A)$. Deduce that A is always diagonalizable over F.

VI. Let V, W be finite-dimensional vector spaces over an algebraically closed field F. Let A, B be linear mappings $A : V \to V$ and $B : W \to W$. Suppose the characteristic polynomials of A and B factor as $c(A) = \prod_i (x - \alpha_i)^{n_i}$ and $c(B) = \prod_j (x - \beta_j)^{m_j}$ (where the α_i and β_j are distinct).

A) What is the characteristic polynomial of the mapping $A \oplus B : V \oplus W \to V \oplus W$?

B) If A and B are diagonalizable, what is the characteristic polynomial of the mapping $A \otimes_F B : V \otimes_F W \to V \otimes_F W$?