MATH 400 – Directed Readings in Module and Representation Theory
Final Problem Set
due: By email or hardcopy, no later than Saturday, May 15

Groundrules: You may consult only our text, Dummit and Foote, *Abstract Algebra*. If you use a result from a chapter we have not discussed, please "footnote" it.

I. Let K, K', P, P', M be modules over a ring R. Assume that P and P' are projective R-modules and that we have short exact sequences

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P' \longrightarrow M \longrightarrow 0$$

(with the same M in the right-most nonzero position in both). In this problem you will show that $K \oplus P' \simeq K' \oplus P$ as R-modules.

A. Using the fact that P is projective show that there is a homomorphism $w : P \to P'$ and a diagram

$$\begin{CD} 0 @>>> K @>i>> P @>>> M @>>> 0 \\ @. @. @VVwV @VVidV @. \\ 0 @>>> K' @>>j> P' @>>> M @>>> 0 \end{CD}$$

in which the square commutes.

B. Continuing from part A, show that there is a homomorphism $u : K \to K'$ such that the left square in

$$\begin{CD} 0 @>>> K @>i>> P @>>> M @>>> 0 \\ @. @VVuV @VVwV @VVidV @. \\ 0 @>>> K' @>>j> P' @>>> M @>>> 0 \end{CD}$$

is also commutative.

C. Show that the sequence

$$0 \longrightarrow K \longrightarrow P \oplus K' \longrightarrow P' \longrightarrow 0$$

is exact, where the map $K \longrightarrow P \oplus K'$ is $x \longmapsto (i(x), u(x))$ and the map $P \oplus K' \longrightarrow P'$ is $(y, z) \longmapsto w(y) - j(z)$.

D. Deduce that $K \oplus P' \simeq K' \oplus P$ as R-modules.

II. Let F be a field and let $R = M_{n\times n}(F)$.

A. Show that there is a unique irreducible R-module M, up to isomorphism.

B. Let M be the module from part A. Show that $Hom_R(M, M)$ is isomorphic to F as a ring.

III. Let F be a field and let $R = F[x]/\langle f(x) \rangle$ for some $f(x) \in F[x]$.

A. State and prove a necessary and sufficient condition on $f(x)$ for R to be a semisimple ring.

B. Describe the Wedderburn decomposition of R in the case that R is semisimple. (That is, what are the n_i and the Δ_i in the isomorphism

$$R \simeq M_{n_1 \times n_1}(\Delta_1) \times \cdots \times M_{n_k \times n_k}(\Delta_k)$$

given by Wedderburn's Theorem?)

Read section 19.1, and using the ideas presented there do the following problems:

IV. Show that the character table of any finite group is an invertible $k \times k$ matrix, where $k =$ the number of conjugacy classes, or the number of irreducible characters.

V. Compute the character tables of

A. The group $S_3 \times S_3$.

B. The alternating group A_5.