Mathematics 134 – Intensive Calculus for Science 2
Solutions for Final Examination – May 10, 2006

I.

A) (10) Give a precise statement of the Fundamental Theorem of Calculus (both parts).

Solution: Part 1: Let f be continuous for $a \le x \le b$, and let F be an antiderivative of f. Then $\int_a^b f(x)\, dx = F(b) - F(a)$.
Part 2: Let f be continuous for $a \le x \le b$, and let $F(x) = \int_a^x f(t)\, dt$. Then $F(x)$ is an antiderivative of $f(x)$ (in other words $F'(x) = f(x)$.

B) (15) The following graph shows $y = f(x)$. Let F be the antiderivative of f with $F(0) = 0$ and F continuous. Sketch the graph $y = F(x)$.

Solution: By the given information and the FTC, we have $16 = \int_0^2 f(x)\, dx = F(2) - F(0)$, so $F(2) = 16 + F(0) = 16$. Similarly, $-7 = \int_2^3 f(x)\, dx = F(3) - F(2)$, so $F(3) = -7 + F(2) = -7 + 16 = 9$. The graph of F should be concave up where $f = F'$ is increasing ($0 < x < 1/2$), then concave down the rest of the way to $x = 3$ (since $f = F'$ is decreasing). F has a local maximum at $x = 2$ (a rather "flat" one).

II. Compute each of the following integrals. You may use the table of integrals anywhere on these. If you do, say which table entry you are using.

A) (10) $\int 3x^6 - 4\sqrt{x} + \sin(x)\, dx$

Solution: By basic rules, the integral is

$$\frac{3}{7}x^7 - \frac{8}{3}x^{3/2} - \cos(x) + C$$

B) (10) $\int \frac{x^3}{x^4+1}\, dx$

Solution: Let $u = x^4 + 1$. Then $du = 4x^3\, dx$, so the form is

$$\frac{1}{4}\int \frac{1}{u}\, du = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|x^4 + 1| + C$$

C) (15) $\int x^2 e^{-9x}\, dx$

Solution: Integrating by parts twice (or using # 14 in the table with $p(x) = x^2$ and $a = -9$), we have

$$\begin{aligned}\int x^2 e^{-9x}\, dx &= \frac{-x^2}{9}e^{-9x} + \frac{1}{9}\int 2xe^{-9x}\, dx\\ &= \frac{-x^2}{9}e^{-9x} + \frac{1}{9}\left(\frac{-2x}{9}e^{-9x} + \frac{1}{9}\int 2e^{-9x}\, dx\right)\\ &= \frac{-x^2}{9}e^{-9x} - \frac{2x}{81}e^{-9x} - \frac{2}{729}e^{-9x} + C\end{aligned}$$

D) (10) $\int \frac{1}{\sqrt{4x^2+3^2}}\ dx$

Solution: This equals $\int \frac{1}{\sqrt{(2x)^2+3^2}}\ dx$ Using # 29 in the table, after a preliminary subsitution $u = 2x$ ($du = 2dx$), and $a = 3$:

$$\begin{aligned} &= \frac{1}{2}\int \frac{1}{\sqrt{u^2+3^2}}\ du \\ &= \frac{1}{2}\ln\left|u + \sqrt{u^2+3^2}\right| + C \\ &= \frac{1}{2}\ln\left|2x + \sqrt{4x^2+9}\right| + C \end{aligned}$$

E) (15) $\int \frac{x+1}{x^2+5x+6}\ dx$

Solution: By partial fractions: $\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$. Clearing denominators, $x+1 = A(x+3) + B(x+2)$. Setting $x = -2$, $A = -1$. Setting $x = -3$, $B = 2$. So

$$\int \frac{x+1}{x^2+5x+6}\ dx = \int \frac{-1}{x+2} + \frac{2}{x+3}\ dx = -\ln|x+2| + 2\ln|x+3| + C$$

(This can be checked by # 27 in the table with $a = -2, b = -3, c = d = 1$.)

III. Let R be the region bounded by $y = x$, the x-axis, and $x = 1$, $x = 4$.

A) (15) Find the volume of the solid obtained by rotating R about the line $y = -2$.

Solution: Cross-sections by planes perpendicular to the x-axis are washers with inner radius 2, outer radius $x + 2$, so

$$V = \int_1^4 \pi(x+2)^2 - \pi(2)^2\ dx = \pi\int_1^4 x^2 + 4x\ dx = \pi\left(x^3/3 + 2x^2\right|_1^4 = 51\pi.$$

B) (10) A thin plate has the shape of the region R (x, y in cm) and density $\delta(x) = x^{-1/2}$ grams/cm^2. Find its total mass.

Solution:

$$M = \int_1^4 x^{-1/2}\cdot x\ dx = \int_1^4 x^{1/2}\ dx = \frac{2}{3}x^{3/2}|_1^4 = \frac{14}{3}$$

IV. At a particular location in Natick on the Mass Pike, a sensor was set up to measure the passage of traffic. The measurements made were used to derive a probability density function for the quantity x = time gap between successive cars (in minutes). The results gave the following formula as a good fit for the pdf: $p(x) = 11(1-x)^{10}$ if $0 < x < 1$, and zero otherwise.

A) (10) Show that p satisfies the usual property for a probability density function: $\int_0^1 p(x)\ dx = 1$.

Solution: In the integral $\int_0^1 11(1-x)^{10}\ dx$, let $u = 1-x$ and $du = -dx$. Changing the limits of integration into their u-equivalents, then reversing the limits of integration and introducing another $-$ sign, we get

$$\int_{u=1}^{u=0} -11u^{10}\ du = \int_0^1 11u^{10}\ du = u^{11}|_0^1 = 1.$$

B) (15) What is the probability that the time gap between successive cars is between $x = .1$ minute and $x = .2$ minute?

Solution: Using the same substitution as in part A, this probability is

$$\int_{.1}^{.2} 11(1-x)^{10}\ dx = \int_{u=.9}^{u=.8} -11u^{10}\ du = \int_{u=.8}^{u=.9} 11u^{10}\ du = u^{11}|_{.8}^{.9} \doteq .228$$

V.

A) (10) Using the definition of Taylor polynomials, compute the Taylor polynomial of degree $n = 3$ for $f(x) = \sqrt{1+2x}$ at $a = 0$.

Solution: We have

$$\begin{aligned} & f(0) = 1 \\ f'(x) &= \frac{1}{2}(1+2x)^{-1/2}\cdot 2 = (1+2x)^{-1/2} \Rightarrow f'(0) = 1 \\ f''(x) &= \frac{-1}{2}(1+2x)^{-3/2}\cdot 2 = -(1+2x)^{-3/2} \Rightarrow f''(0) = -1 \\ f'''(x) &= \frac{3}{2}(1+2x)^{-5/2}\cdot 2 = 3(1+2x)^{-1/2} \Rightarrow f'''(0) = 3 \end{aligned}$$

So the Taylor polynomial of degree 3 is

$$1 + x - \frac{1}{2}x^2 + \frac{3}{6}x^3 = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$$

B) (10) Use our shortcut methods to check your work in part A.

Solution: Substituting $u = 2x$ in the binomial series with $p = 1/2$:

$$1 + \frac{1}{2}(2x) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2}(2x)^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{6}(2x)^3 = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$$

C) (5) Use your polynomial from part A to compute an approximation to $\sqrt{1.2}$. What is the error in your approximation?

Solution: We take $x = .1$ so

$$\sqrt{1.2} = \sqrt{1 + 2(.1)} \doteq 1 + (.1) - (.1)^2/2 + (.1)^3/3 = 1.09550$$

Using a calculator, the exact value is about 1.09544, so the error is about .00006.

VI. (15) Solve for y by separation of variables: $\frac{dy}{dx} = \cos(x)(1 + y^2)$ with $y(0) = 1$.

Solution: After separation, $\frac{dy}{1+y^2} = \cos(x)\ dx$. Integrating both sides, we get

$$\arctan(y) = \sin(x) + C \Rightarrow y = \tan(\sin(x) + C).$$

Then from the initial condition, $1 = \tan(\sin(0) + C)$, so $C = \pi/4$. The solution is

$$y = \tan(\sin(x) + \pi/4)$$

VII. An avian flu epidemic has broken out in Birdsburgh, a large city with total population 10 million. Write N for the number of people who have been infected, as a function of time. The Birdsburgh Public Health department determines that:

$$\frac{dN}{dt} = kN(10 - N), \tag{1}$$

or in words: the rate of change of N is proportional to the product of N and the number of people not yet infected, where N is in millions of people, t in weeks, k a positive constant.

A) (10) Which of the following slope field plots matches (1)? Explain how you can tell.

Solution: The one that matches is slope field B. Note that A has an equilibrium at $N = 1$ which doesn't match the equation. (Or, note that since it is given $k > 0$, the slope values should be positive for $0 < N < 10$. Only B satisfies that.)

B) (10) For what value of k is $N(t) = 10/(1 + 1000e^{-t})$ a solution of (1)?

Solution: For this $N(t)$, by the chain rule

$$\frac{dN}{dt} = \frac{10000e^{-t}}{(1 + 1000e^{-t})^2}$$

The right side is

$$\begin{aligned} kN(10 - N) &= k \cdot \frac{10}{1 + 1000e^{-t}} \cdot \left(10 - \frac{10}{1 + 1000e^{-t}}\right) \\ &= k \cdot \frac{10}{1 + 1000e^{-t}} \cdot \frac{10000e^{-t}}{1 + 1000e^{-t}} \\ &= \frac{100000ke^{-t}}{(1 + 1000e^{-t})^2} \end{aligned}$$

Comparing the two, we see $100000k = 10000$, so $k = .1$.

C) (5) If the epidemic proceeds according to the function $N(t)$ in part B, how many weeks will pass before the number of infected people reaches 1 million?

Solution: We solve

$$1 = 10/(1 + 1000e^{-t}) \Rightarrow 1 + 1000e^{-t} = 10 \Rightarrow e^{-t} = 9/1000$$

So $t = -\ln(9/1000) = 4.71053$ weeks.

Extra Credit (20) The hull of a boat is 20 feet long. At a distance s feet from the bow (the front), the cross section of the part of the hull below $y = 0$ (the waterline) has the shape of the region in the xy-plane below $y = 0$ and above the parabola $y = ax^2 - b$, where a, b are given in the following table:

s	5	10	15	20
a	2	3	4	5
b	2	3	4	4

Estimate the *volume* enclosed by the hull below the water line.

Solution: Think of the usual approach for computing volumes by slicing: $V = \int A(s)\,ds$. Since we only have the information at 5-foot intervals, we will compute $A(s)$ for $s = 5, 10, 15$ and use a left-hand sum approximation for the integral of $A(s)$. $A(5)$ is the area between $y = 2x^2 - 2$ and the x-axis:

$$A(5) = \int_{-1}^{1} 2x^2 - 2\,dx = -8/3$$

Then $A(10)$ is the area between $y = 3x^2 - 3$ and the x-axis:

$$A(10) = \int_{-1}^{1} 3x^2 - 3\,dx = -4$$

And $A(15)$ is the area between $y = 4x^2 - 4$ and the x-axis:

$$A(15) = \int_{-1}^{1} 4x^2 - 4\,dx = -16/3$$

The left-hand sum approximation for the volume of the hull below the waterline is

$$V \doteq (8/3) \cdot 5 + (4) \cdot 5 + (16/3) \cdot 5 = 60$$

(cubic feet). (Note: we got rid of the signs in the answers for $A(5)$, etc. because the negatives just tell us the area is below the x-axis.)