Mathematics 134 – Intensive Calculus for Science 2
Solutions for Exam 2 – March 24, 2006

I. Compute each of the integrals below, using some combination of integration by substitution, parts, and partial fractions. *You must show all work for full credit. Applying an entry from the integral table is OK if necessary, but that will result in a deduction of 3 points per part.*

A) Use integration by parts with $u = x$, $dv = \cos(4x)\ dx$. Then $du = dx$, $v = \frac{1}{4}\sin(4x)$ and

$$\int x\cos(4x)\ dx = \frac{x}{4}\sin(4x) - \int \frac{1}{4}\sin(4x)\ dx = \frac{x}{4}\sin(4x) + \frac{1}{16}\cos(4x) + C$$

B) By partial fractions: The degree of the top is already strictly smaller than the degree of the bottom so we do not need to divide. Then we factor $x^2 - 6x + 8 = (x-2)(x-4)$ and the partial fractions are:

$$\frac{x+3}{x^2 - 6x + 8} = \frac{A}{x-2} + \frac{B}{x-4}.$$

We clear denominators to get

$$x + 3 = A(x-4) + B(x-2).$$

Setting $x = 2$ gives $5 = -2A$, so $A = -5/2$. Then setting $x = 4$ gives $7 = 2B$, so $B = 7/2$.

$$\int \frac{-5/2}{x-2} + \frac{7/2}{x-4}\ dx = \frac{-5}{2}\ln|x-2| + \frac{7}{2}\ln|x-4| + C$$

II. Compute each of the following integrals using an entry or entries from the table of integrals. Say which entry or entries you are using.

A) Use #17 with $n = 4$, and then again with $n = 2$:

$$\int \sin^4(x)\ dx = \frac{-1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\cos(x)\sin(x) + \frac{3x}{8} + C$$

B) Use #14 with $p(x) = x^2 + 12$ and $a = 3$:

$$\int (x^2 + 12)e^{3x}\ dx = \frac{1}{3}(x^2 + 12)e^{3x} - \frac{2x}{9}e^{3x} + \frac{2}{27}e^{3x} + C$$

III. (20) Let R be the region bounded by $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 1$. Find the volume of the solid obtained if R is rotated about the line $y = 5$.

Solution: The parabola $y = x^2 + 2$ lies above the line $y = x$ for all x. But there is a gap between the region R and the line $y = 5$ (the axis of rotation). So the cross-sections of the solid perpendicular to the x-axis are *washers* with inner radius $5 - (x^2 + 2) = 3 - x^2$ and outer radius $5 - x$. The volume is the integral of the area of the cross-section:

$$\begin{aligned} V &= \int_0^1 \pi(5-x)^2 - \pi(3-x^2)^2 \, dx \\ &= \pi \int_0^1 16 - 10x + 7x^2 - x^4 \, dx \\ &= \pi \left(16x - 5x^2 + \frac{7}{3}x^3 - \frac{x^5}{5} \right|_0^1 \\ &= \frac{197\pi}{15} \end{aligned}$$

IV. A long thin straight wire extends from $x = 0$ to $x = 1$ cm. The density of the wire at x is $\delta(x) = 2 + \cos(\pi x)$ grams per cm.

A) (10) Set up, *but do not evaluate* the formula computing the *center of mass* of the wire.

Solution: Note the problem says "long straight" wire – think of it as a segment of the x-axis. The center of mass is at

$$\overline{x} = \frac{\int_0^1 x(2 + \cos(\pi x)) \, dx}{\int_0^1 2 + \cos(\pi x) \, dx}$$

(Note: the $2 + \cos(\pi x)$ is the density of the metal in the wire, not the *shape* of the wire!)

B) (10) If the midpoint Riemann sum with $n = 10$ is used to approximate the integral for the total mass, will the result be an overestimate or an underestimate? Explain.

Solution: The mass integral is

$$M = \int_0^1 \cos(\pi x) + 2 \, dx.$$

The function $\cos(\pi x) + 2$ has a graph that is concave down from $x = 0$ to $x = 1/2$, then concave up from $x = 1/2$ to $x = 1$. By the symmetry of the cosine graph, we can see that the overestimate areas on the first half are exactly balanced by the underestimate areas on the second half. So the correct answer is: *neither* – the midpoint Riemann sum gives the *exact value*(!)

Scoring note: This was a tricky question and no one in the class actually got it correct (for the correct reason). So what I did to score this problem was:

- If you said the answer depended on the concavity of the graph $y = \cos(\pi x) + 2$ and got the estimates going the right way (that is, overestimate where concave down, underestimate where concave up), then I gave full credit (10 points)
- If you said the answer depended on the concavity of the graph $y = \cos(\pi x) + 2$ and got the estimates going the wrong way, then I gave part credit (8 points)
- If you didn't make the connection with concavity, instead of giving no credit, I *dropped the question entirely*, and computed your grade using the other 90 possible points.

(This seemed the fairest way to do it and it actually helped everyone in all three categories.)

V. (10) The graph $y = f(x)$ below shows either a probability density (pdf) or a cumulative distribution (cdf). Say which it is, and find the value of c given in the graph.

Solution: The given graph is a pdf because the total area between the graph is finite and the function is not always increasing (and does not approach 1 as $x \to +\infty$). The value of c is determined from the requirement that the total area is 1. Using area formula for rectangles: $1 = 2(2c) + c = 5c$, so $c = 1/5$.