Mathematics 133 – Intensive Calculus for Science 1
Sample Exam II Solutions – October 29, 2005

I. Refer to the graph on the handout showing *the derivative* $y = f'(x)$ for some function $f(x)$.

A) Using the information here, construct a "qualitative" plot of $y = f''(x)$.

Solution: $f''(x)$ is the derivative of $f'(x)$, so the y-coordinates of your graph should show the slopes of the given graph. $f''(x) = 0$ for all $x < 1$ since the given graph is horizontal on that interval. f'' starts close to zero for $x > 1$, increases up to a positive maximum around $x = 2$, decreases to zero a bit after $x = 3$, then is negative the rest of the way. It reaches its negative minimum around $x = 4$, then increases at bit after that (but still stays negative).

B) Over which intervals is f increasing?

Solution: Since f is increasing where f' is positive, f is increasing on the intervals where the y-coordinates of the given graph are positive: $x < 1$ and $2 < x < 4$.

C) Is f' continuous at $x = 1$? Why or why not? What happens on the graph $y = f(x)$ at $x = 1$?

Solution: f' is not continuous at $x = 1$ because of the "jump" in the given graph. We have $\lim_{x\to 1^-} f'(x) = 2$ but $\lim_{x\to 1^+} f'(x) = -1$. This indicates that the slope of the graph $y = f(x)$ changes suddenly from $+2$ to -1 at $x = 1$. Assuming f is continuous at $x = 1$, its graph would have a "sharp corner" at $x = 1$.

II.

A) The function $H(t)$ gives the number of hours of daylight t days after the start of the year in Worcester. At $t = 304$ days (October 31 in a non-leap year), $H'(304) = -0.083$. Give the meaning of this equation as a sentence, using appropriate units.

Solution: At $t = 304$ (that is, on October 31), the number of hours of daylight per day in Worcester is decreasing at the rate of 0.083 hours per day (about 5 minutes per day). (Note: It's decreasing because the derivative value is negative.)

B) The table below gives the position s (in miles) of a freight train moving along a straight line track as a function of time t (in hours).

t	.5	1	1.5	2	2.5
s	10	25	42	50	55

Estimate the train's instantaneous velocity at $t = 1.5$ hours as closely as you can from this information.

Solution: From the table information, we can estimate the instantaneous velocity by average velocities in many ways. Thinking of the definition of the instantaneous velocity as a limit, we expect the closest approximations would be for $h = \Delta t$ as small as possible:

$$v \doteq \frac{s(2) - s(1.5)}{2 - 1.5} = \frac{50 - 42}{2 - 1.5} = 16$$

or

$$v \doteq \frac{s(1) - s(1.5)}{1 - 1.5} = \frac{25 - 42}{1 - 1.5} = 34$$

The actual instantaneous velocity at $t = 1.5$ is probably somewhere in between, so the *average* is our best estimate: $v \doteq \frac{16+34}{2} = 25$ miles per hour.

III. Using the limit definition, find $f'(x)$ for $f(x) = 1/x$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{x - (x+h)}{(x+h)xh} \\ &= \lim_{h\to 0} \frac{-h}{(x+h)xh} \\ &= \lim_{h\to 0} \frac{-1}{(x+h)x} \\ &= \frac{-1}{x^2} \end{aligned}$$

(Check with the power rule: $\frac{d}{dx}x^{-1} = -x^{-2} = \frac{-1}{x^2}$.)

IV. Find derivatives of each of the following functions by applying the appropriate "short-cut" derivative rules:

A) $f(x) = 5x^7 - \frac{3}{\sqrt{x}} - 4^{2x}$

Solution: $f'(x) = 35x^6 + \frac{3}{2}x^{-3/2} - 4^{2x}\ln(4) \cdot 2$

B) $g(x) = (x^2+1)^{12}2^x$

Solution: By the product and chain rules:

$$g'(x) = 12(x^2+1)^{11}2^x + (x^2+1)^{12}2^x\ln(2) = (x^2+1)^{11}2^x(12 + (x^2+1)\ln(2))$$

C) $h(x) = \frac{x^2}{e^x - 1}$

Solution: By the quotient rule:

$$h'(x) = \frac{(e^x - 1)(2x) - x^2 e^x}{(e^x-1)^2}$$

V. (5) Say whether the following statement is true or false, and explain your reasoning: If the time interval is short enough, then we expect the average velocity of a car over the interval will be close to its instantaneous velocity at any time in the interval.

Solution: True: Since $v_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$, the instantaneous velocity at each t_0 is the limit of average velocities as the length of the time interval goes to zero. If the time interval is short enough, we would expect the average velocity over the whole interval to be very close to the limiting value. (Of course I am thinking of a "usual" physical situation where the position and the speed are changing in a continuous fashion, so that even if the time t_0 where the instantaneous velocity is computed is not at one of the endpoints of the time interval, the average velocity over the whole interval would still be close to the average velocities over the smaller intervals that would be used to compute $v_{\text{inst}}(t_0)$.)