Mathematics 133 – Intensive Calculus for Science 1
Final Examination – Solutions
December 13, 2005

I. $f(x)$ is approximately exponential. With $f(x) = ca^x$, we have

$$ca^{.3} = 4.12$$
$$ca^{.6} = 3.74$$

So dividing, $a^{.3} \doteq .908$ and $a \doteq (.908)^{10/3} = .724$. Then $c \doteq 4.12/(.724)^{.3} = 4.54$:

$$f(x) \doteq 4.54(.724)^x.$$

$g(x)$ is linear since the slope is constant: $\frac{g(.6)-g(.3)}{.6-.3} = -.8$ so $g(x) = (-.8)(x - .3) + 4.31 = -.8x + 4.55$:

$$g(x) = -.8x + 4.55.$$

II. A) The graph is sinusoidal with period 4, amplitude 5 and vertical shift +3:

$$f(x) = 5\sin\left(\frac{2\pi x}{4}\right) + 3 = 5\sin\left(\frac{\pi x}{2}\right) + 3$$

B) The function has an inverse function if we restrict to $-1 \le x \le 1$ because that portion of the graph passes the horizontal line test: horizontal lines $y = k$ for $-2 \le k \le 8$ intersect that section of the graph exactly once. The domain of the inverse function is $-2 \le x \le 8$.

III.
A)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{5(x+h)^2 - (x+h) + 3 - 5x^2 + x - 3}{h} \\ &= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - x - h + 3 - 5x^2 + x - 3}{h} \\ &= \lim_{h\to 0} 10x - 1 + 5h \\ &= 10x - 1. \end{aligned}$$

B) By the chain rule and the rule for exponential functions:

$$g'(x) = 3^x \ln(3) - \frac{1}{x^3 + x} \cdot (3x^2 + 1) = 3^x \ln(3) - \frac{3x^2 + 1}{x^3 + x}$$

C) Rewrite $h(x)$ using powers: $h(x) = 3x^{1/5} + 7x^{-1/3} + 3x^8$. Then

$$h'(x) = \frac{3}{5}x^{-4/5} - \frac{7}{3}x^{-4/3} + 24x^7$$

D) By the quotient rule:

$$i'(x) = \frac{(\sin^2(x) - 1)(-\sin(x)) - \cos(x) \cdot 2\sin(x)\cos(x)}{(\sin^2(x) - 1)^2}$$

E)

$$j'(x) = \frac{1}{1 + (3x+2)^2} \cdot 3$$

F) Using implicit differentiation:

$$2y\frac{dy}{dx}\sqrt{x} + y^2\frac{1}{2\sqrt{x}} - e^y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-y^2\frac{1}{2\sqrt{x}}}{2y\sqrt{x} - e^y} = \frac{-y^2}{4xy - 2\sqrt{x}e^y}$$

IV.

A) f is increasing on intervals where $f'(x) \geq 0$, so here for all $x > -.5$. f is decreasing where $f'(x) < 0$, so $x < -.5$.

B) $y = f(x)$ is concave up on intervals where $f''(x) > 0$, or equivalently, where $f'(x)$ is increasing: $x < 0$ and $x > 1$. $y = f(x)$ is concave down on intervals where $f'(x)$ is decreasing: $0 < x < 1$.

C) The graph should be parabolic in shape with x-axis intercepts at $x = 0, x = 1$, opening up. The vertex of the parabola should be at $x = .5$.

V. A) The temperature at time $t = 0$ is $H(0) = 70 - 50 = 20$ degrees.

B) The rate of change of the temperature at time $t = 10$ is $H'(10) = 5e^{-10/10} \doteq 1.84$ degrees F per minute.

C) The temperature reaches 60 degrees when

$$60 = 70 - 50e^{-t/10} \Rightarrow \frac{-10}{-50} = e^{\frac{-t}{10}} \Rightarrow t = -10\ln(1/5) \doteq 16.1$$

minutes.

VI. By the product and chain rules, $f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x) = e^{-x}(\cos(x) - \sin(x))$. For $0 \leq x \leq \pi$, the only solution of $f'(x) = 0$ is $x = \pi/4$. Then

$$f(0) = 0 \qquad f(\pi/4) = e^{-\pi/4}\sin(\pi/4) \doteq .322 \qquad f(\pi) = 0$$

so the minimum value is 0 and the maximum value is $e^{-\pi/4}\sin(\pi/4)$.

VII. A) The cost to build the pipeline is 3 times the distance under water, plus 2 times the distance along the shore:

$$C(x) = 3\sqrt{4+x^2} + 2(6-x)$$

B) We find the critical points of C:

$$C'(x) = \frac{3}{2}(4+x^2)^{-1/2}(2x) - 2 = \frac{3x}{\sqrt{x^2+4}} - 2$$

This is zero when

$$3x = 2\sqrt{x^2+4} \Rightarrow 9x^2 = 4x^2 + 16 \Rightarrow 5x^2 = 16$$

So $x = \frac{4}{\sqrt{5}} \doteq 1.79$ (ignore the negative root since it clearly cannot give a minimium of the cost!). This is a minimum by the 1st derivative test: $C'(1) = \frac{3}{\sqrt{5}} - 2 \doteq -.66 < 0$ while $C'(2) = \frac{6}{\sqrt{8}} - 2 \doteq .12 > 0$. The pipeline of minimum cost goes from the island to the point with $x = 1.79$ miles along the shore.

VIII. The volume of the cube is $V = x^3$ where x is the length of the side. So

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

when $V = 216$, $x = \sqrt[3]{216} = 6$, and $\frac{dV}{dt} = -10$, so

$$-10 = 3(6)^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{-10}{108} = \frac{-5}{54}$$

The side of the cube is decreasing at $\doteq .093$ cm/min.