Mathematics 133 – Intensive Calculus for Science 1
Solutions for Midterm Exam 3 – December 2, 2005

I. $f(x) = xe^{-x^2}$.

A) By the product and chain rules:

$$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = (1 - 2x^2)e^{-x^2}.$$

B) The critical points of f occur when $1 - 2x^2 = 0$, so $x = \frac{\pm 1}{\sqrt{2}} = \frac{\pm\sqrt{2}}{2} \doteq \pm .707$. Of these, only $-\sqrt{2}/2 \doteq -.707$ is in our interval. Evaluating f at the critical point and at the endpoints:

$$\begin{aligned} f(-2) &= -2e^{-4} \doteq -.0366 \\ f(-\sqrt{2}/2) &= -\sqrt{2}/2 \cdot e^{-1/2} \doteq -.4289 \\ f(0) &= 0 \end{aligned}$$

So the maximum value is $f(0) = 0$ and the minimum is $f(-\sqrt{2}/2) = -\sqrt{2}/2 \cdot e^{-1/2} \doteq -.4289$.

C) Yes it does. From the form of f' we can see that f' changes from negative to positive at $-\sqrt{2}/2$, so by the First Derivative Test, this is a local minimum.

II.

A) By the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(\cos(x) + \sin(x))\cos(x) - \sin(x)(-\sin(x) + \cos(x))}{(\sin(x) + \cos(x))^2} \\ &= \frac{\cos^2(x) + \sin(x)\cos(x) + \sin^2(x) - \sin(x)\cos(x)}{(\sin(x) + \cos(x))^2} \\ &= \frac{1}{(\sin(x) + \cos(x))^2} \end{aligned}$$

B) By the rule for natural log plus the chain rule:

$$g'(x) = \frac{1}{3x + \frac{1}{x}}(3 - \frac{1}{x^2}) = \frac{3x^2 - 1}{3x^3 + x}$$

C) By the rule for arcsin and the chain rule:

$$h'(x) = \frac{1}{\sqrt{1 - (4x)^2}} \cdot 4 = \frac{4}{\sqrt{1 - 16x^2}}.$$

D) By implicit differentiation (using product rule):

$$y^2 + 2xy\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$(2xy + x^2)\frac{dy}{dx} = -2xy - y^2,$$

so

$$\frac{dy}{dx} = \frac{-2xy - y^2}{2xy + x^2}.$$

III. Call the dimensions of the rectangle x, y (in yards). Say x is the side parallel to the existing fence. Then we have $xy = 500$ and we want to minimize $L = x + 2y = x + \frac{1000}{x}$. Then $L' = 1 - \frac{1000}{x^2} = 0$ when $x = \sqrt{1000} = 10\sqrt{10}$. This is a minimum, since it is the only critical point with positive x, and $L'' = \frac{1000}{x^3}$, so $L'' > 0$ for all $x > 0$. The dimensions of the lot with minimum fence are $x = 10\sqrt{10} \doteq 31.6$ and $y = 5\sqrt{10} \doteq 15.8$ (from $xy = 5000$).

IV. For the sphere, $V = \frac{4\pi}{3}r^3$. So taking time derivatives:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

We are given $\frac{dr}{dt} = 3$, so when $r = 6$,

$$\frac{dV}{dt} = 4\pi \cdot (6)^2 \cdot 3 = 432\pi$$

(units are $\mathrm{mm}^3/\mathrm{min}$).