Mathematics 133 – Intensive Calculus for Science 1
Exam 1 Solutions
October 3, 2005

I. First note that f cannot be either linear or exponential because it is increasing for some x and decreasing for other x.

A) $g(x)$ is the only linear function, because it is the only one with constant slope. Using the first two points for instance $m = \frac{28-23}{0-(-2)} = \frac{5}{2}$. The y-intercept is 28 from the table so $g(x) = \frac{5}{2}x + 28$.

B) $h(x)$ is exponential. This can be seen because the ratios of successive values is constant:

$$\frac{18}{2} = \frac{2}{2/9} = \frac{2/9}{2/81} = \frac{2/81}{2/729} = 9.$$

To find the formula, we try to determine a, b to match the table with $h(x) = ba^x$. From $h(0) = 2$, we get $2 = ba^0 = b$. Then from $h(-2) = 18$ we get $18 = 2a^{-2}$ so $9 = a^{-2}$, so $a^2 = 1/9$, and $a = 1/3$. The formula is $h(x) = 2\left(\frac{1}{3}\right)^x$.

II. Because the "Explosium" decays exponentially, we know the amount is given by a formula $Q(t) = Q_0e^{kt}$ or $Q(t) = Q_0a^t$. Either form can be used, and we will show how to do the problem using both of them.

Method 1. Using $Q(t) = Q_0e^{kt}$, the initial amount is $Q_0 = 10$, and the half-life is 12 years so

$$\begin{aligned} 5 &= 10e^{k\cdot 12} \\ 1/2 &= e^{12k} \\ \ln(1/2) &= 12k \\ \frac{\ln(1/2)}{12} &= k \end{aligned}$$

Then we want to determine the time t when $Q(t) = 1$:

$$\begin{aligned} 1 &= 10e^{\frac{\ln(1/2)}{12}t} \\ 1/10 &= e^{\frac{\ln(1/2)}{12}t} \\ \ln(1/10) &= \frac{\ln(1/2)}{12}t \\ \frac{12\ln(1/10)}{\ln(1/2)} &= t \end{aligned}$$

Method 2. Using $Q(t) = Q_0a^t$, we have $Q = 10a^t$ and substituting $t = 12$ (the half-life) we get $5 = 10a^{12}$, so $a = (1/2)^{\frac{1}{12}}$ and $Q = 10\left(\left(\frac{1}{2}\right)^{\frac{1}{12}}\right)^t = 10\left(\frac{1}{2}\right)^{t/12}$. Then as before we want to solve for t in the equation $1 = 10\left(\frac{1}{2}\right)^{t/12}$ We get:

$$1/10 = \left(\frac{1}{2}\right)^{t/12}$$
$$\ln(1/10) = (t/12)\ln(1/2)$$
$$\frac{12\ln(1/10)}{\ln(1/2)} = t$$

(which agrees with the other method, of course!)

III. A) Because this plot is sinusoidal, with a maximum at $x = 0$, it is simplest to use the form $y = A\cos(Bx) + C$. The amplitude $A = (5 - (-3))/2 = 4$. The period is 1, so $2\pi/B = 1$ and $B = 2\pi$. The graph is also shifted vertically by $C = -1$:

$$y = 4\cos(2\pi x) - 1.$$

B) This graph looks like a polynomial of degree 3 since there are 2 turning points, a double root at $x = 1$, and a root at $x = 2$. This gives $y = k(x-1)^2(x-2)$ for some constant k. Since the graph "starts high and finishes low" we know $k < 0$. The exact value can be determined because the graph shows the y-intercept at $y = 6$:

$$6 = k(0-1)^2(0-2) = -2k$$

So $k = -3$ and the formula is

$$y = -3(x-1)^2(x-2).$$

IV. The graph $y = 10^{x+1}$ is an exponential growth curve (shifted left by 1). So it does pass the horizontal line test (horizontal lines $y = c > 0$ intersect the graph exactly once). This says there is an inverse function. To find the formula, interchange x and y then solve for y taking logarithms:

$$x = 10^{y+1}$$
$$\ln(x) = (y+1)\ln(10)$$
$$\frac{\ln(x)}{\ln(10)} = y + 1$$
$$\frac{\ln(x)}{\ln(10)} - 1 = y$$

The inverse function is $f^{-1}(x) = \frac{\ln(x)}{\ln(10)} - 1$.

V. A) $r(x)$ has vertical asymptotes where the *bottom* is zero: $16 - x^2 = 0$, so $x = \pm 4$.
B) $r(x)$ has a horizontal asymptote at $y = -2$ because as $x \to \pm\infty$,

$$\frac{2x^2 - x}{16 - x^2} = \frac{2 - \frac{1}{x}}{\frac{16}{x^2} - 1} \to -2$$

C) The graph $y = r(x)$ crosses the x-axis where the *top* is zero: $0 = 2x^2 - x = x(2x-1)$, so $x = 0$ or $x = 1/2$.