MATH 392 – Geometry Through History
Group Discussion – One testimony about the impact of Euclid, January 29

In his brief biography of the philosopher Thomas Hobbes from about 1690 CE, John Aubrey wrote:

He [that is, Hobbes] was . . . 40 yeares old before he looked on geometry; which happened accidentally. Being in a gentleman's library in . . . , Euclid's Elements lay open, and 'twas the 47 El. libri I [that is, the 47th proposition in Book I]. He read the proposition. 'By G–,' (he would now and then sweare an emphaticall Oath by way of emphasis) sayd he, 'this is impossible!' So he reads the demonstration of it, which referred him back to such a proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps *that at last he was demonstratively convinced of that trueth. This made him in love with geometry.*

Today, we want to recreate something like the experience that Aubrey is referring to by reading through the proof that Euclid gives for his Proposition 47 in Book I of the *Elements* and seeing how the results laid down previously prepare for this surprising and elegant proof. Here's the statement and the proof as Euclid gives it. (Notes: I have written this using some modern notation for quantities such as angles. The translation from the original Greek is based on the standard version by T.L. Heath, with some slight "tweaks" by J.L. to show the literal wording more closely in a few instances):

Proposition 47. In right-angled triangles the square on the side opposite the right angle is equal to the squares on the sides containing the right angle.

Questions, part 1: What is the usual name we give to this result and how do we usually state it? Euclid's point of view is different because he essentially does not have the "infrastructure" of algebra–it had literally not yet been invented in his time*(!)–and he wants to state everything in a geometric form. What sort of equality is this asserting (i.e. what two quantities are equal)? Note: Some of this may only become clear as you work through the rest of the questions below. If you don't see it now, come back later(!)*

Note: Refer to the figure at the top of the next page for all notation.

Proof: Let ABC be a right-angled triangle having the angle $\angle BAC$ right.

I say that the square on BC is equal to the squares on BA and AC.

Let the square $BDEC$ on the side BC have been described, and [similarly] the squares GB and HC on BA and AC (Proposition 46).

Questions, part 2: Look at the list of propositions in Book I given in McCleary's book, pages 24-26. What does Proposition 46 say? Can you guess how that construction works and which of the earlier Propositions it relies on?

Let the line AL through A have been constructed parallel to either BD or CE, and let AD and FC have been joined (Proposition 31, Postulate 1).

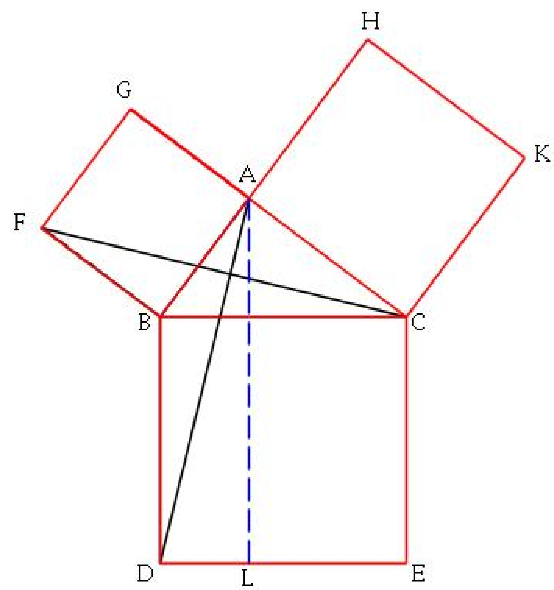

Figure 1: Figure for Proposition 47, Book I

Questions, part 3: Again look at the list of propositions in Book I given in McCleary's book, pages 24-26 and the discussion of the postulates in the preceding section of the text. What does Proposition 31 say? Do you know a way to construct that parallel line? What does Postulate 1 say?

Since each of the angles BAC and BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC and AG not lying on the same side make the adjacent angles equal to two right angles, therefore CA is in a straight line with AG. For the same reason BA is also in a straight line with AH (Proposition 14).

Questions, part 4: What does Proposition 14 say? Note: Euclid doesn't use any numerical measure for angles so he needs to work a bit harder. How could we derive the statement in Proposition 14 more simply?

Since the angle $\angle DBC$ equals the angle $\angle FBA$, for each is right, let the angle $\angle ABC$ have been added to each, therefore the whole angle $\angle DBA$ equals the whole angle $\angle FBC$ (Postulate 4, Common Notion 2).

Questions, part 5: What do Postulate 4 and Common Notion 2 say?

Since DB equals BC, and FB equals BA, the two sides AB and BD equal the two sides FB and BC respectively, and the angle $\angle ABD$ equals the angle $\angle FBC$, therefore the base AD equals the base FC, and the triangle ΔABD equals the triangle ΔFBC (Proposition 4).

Questions, part 6: What does Proposition 4 say? What is our usual name for that fact? What does it mean here to say the triangle ΔABD equals the triangle ΔFBC? They're clearly not exactly the same triangle, so what kind of relation is this equality?

Now the parallelogram BL is double the triangle ΔABD, for they have the same base BD and are in the same parallels BD and AL. And the square GB is double the triangle FBC, for they again have the same base FB and are in the same parallels FB and GC (Proposition 41).

Questions, part 7: What does Proposition 41 say? How would we usually say this sort of thing using standard facts from elementary geometry today? And to what does the "double" refer here? What is being doubled?

Therefore the parallelogram BL also equals the square GB (Common Notion 1).

Questions, part 8: How is the use of "equals" here different from the use of the same word before part 6 of the questions above? Explain the two different senses of that word.

Similarly, if AE and BK are joined, it will be shown that the parallelogram CL is equal to the square HC. Therefore the whole square $BDEC$ equals the sum of the two squares GB and HC (Common Notion 2).

And the square $BDEC$ is described on BC, and the squares GB and HC on BA and AC.

Therefore the square on BC equals the sum of the squares on BA and AC.

Therefore in right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle, which is what it was necessary to show.

Questions, part 9: Read over the Aubrey quotation about Hobbes's experience again. Do you see now what he was getting at?

What kind of thinker is likely to get really excited about this sort of demonstration of the truth of a statement? Some food for thought–Albert Einstein once said:

We reverence ancient Greece as the cradle of western science. Here for the first time the world witnessed the miracle of a logical system which proceeded from step to step with such precision that every single one of its propositions was absolutely indubitable—I refer to Euclid's geometry. This admirable triumph of reasoning gave the human intellect the necessary confidence in itself for its subsequent achievements. If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker.