

MATH 357 -- Combinatorics  
 A Computation With the Cycle Index Polynomial  
 April 21, 2017

Say  $G = C_8$  acting on the wheel with 8 wedge-shaped sectors. As we just saw

$$Z := \frac{(x1^8 + x2^4 + 2 \cdot x4^2 + 4 \cdot x8)}{8};$$

$$\frac{1}{8} x1^8 + \frac{1}{8} x2^4 + \frac{1}{4} x4^2 + \frac{1}{2} x8 \quad (1)$$

By the ***Polya Enumeration Theorem***, to find the number of distinct colorings of the sectors using 3 colors we compute

$$Z_{sub} := \text{subs}(\{x1 = c1 + c2 + c3, x2 = c1^2 + c2^2 + c3^2, x4 = c1^4 + c2^4 + c3^4, x8 = c1^8 + c2^8 + c3^8\}, Z);$$

$$\frac{1}{8} (c1 + c2 + c3)^8 + \frac{1}{8} (c1^2 + c2^2 + c3^2)^4 + \frac{1}{4} (c1^4 + c2^4 + c3^4)^2 + \frac{1}{2} c1^8 + \frac{1}{2} c2^8 + \frac{1}{2} c3^8 \quad (2)$$

$\text{expand}(Z_{sub});$

$$c1^8 + c1^7 c2 + c1^7 c3 + 4 c1^6 c2^2 + 7 c1^6 c2 c3 + 4 c1^6 c3^2 + 7 c1^5 c2^3 + 21 c1^5 c2^2 c3 + 21 c1^5 c2 c3^2 + 7 c1^5 c3^3 + 10 c1^4 c2^4 + 35 c1^4 c2^3 c3 + 54 c1^4 c2^2 c3^2 + 35 c1^4 c2 c3^3 + 10 c1^4 c3^4 + 7 c1^3 c2^5 + 35 c1^3 c2^4 c3 + 70 c1^3 c2^3 c3^2 + 70 c1^3 c2^2 c3^3 + 35 c1^3 c2 c3^4 + 7 c1^3 c3^5 + 4 c1^2 c2^6 + 21 c1^2 c2^5 c3 + 54 c1^2 c2^4 c3^2 + 70 c1^2 c2^3 c3^3 + 54 c1^2 c2^2 c3^4 + 21 c1^2 c2 c3^5 + 4 c1^2 c3^6 + c1 c2^7 + 7 c1 c2^6 c3 + 21 c1 c2^5 c3^2 + 35 c1 c2^4 c3^3 + 35 c1 c2^3 c3^4 + 21 c1 c2^2 c3^5 + 7 c1 c2 c3^6 + c1 c3^7 + c2^8 + c2^7 c3 + 4 c2^6 c3^2 + 7 c2^5 c3^3 + 10 c2^4 c3^4 + 7 c2^3 c3^5 + 4 c2^2 c3^6 + c2 c3^7 + c3^8 \quad (3)$$

$$\text{coeff}(\text{coeff}(\text{coeff}(Z_{sub}, c3), c2^4), c1^3) \quad 35 \quad (4)$$

This is the coefficient of  $c1^3 \cdot c2^4 \cdot c3$ , which represents the number of distinct ways to color the wheel using three colors, so that exactly three wedges are colored red (color 1), 4 are colored green (color 2), and 1 is colored black (color 3).

The count in this particular case can be seen by ***other methods too***, of course to check our work. Since there is only one black wedge, we can rotate it to position 1. Then the other 7 wedges must be colored 3 red and 4 green and they are fixed in

positions 2, ..., 8. (We used the initial rotation to "standardize" the positions to be counted and pick one particular element out of each  $C_8$ -equivalence class (one out of each orbit. The number of ways to color the remaining 7 wedges using two colors is  $\binom{7}{3} = \binom{7}{4} = 35$ .

$\text{binomial}(7, 3)$ ;

35

(5)