MATH 392 – Geometry Through History
Discussion – An interesting abstract surface
April 22 and 25, 2016

Background

As we have seen, it's possible to define Riemannian metrics, Gaussian curvature, and geodesics on abstract surfaces once we have a good analog of the tangent space at each point. (But unlike the situation of a surface in $\mathbb{R}^3$, we had to cook up a pretty non-intuitive way to define the elements the elements of the tangent space. They are constructed as directional derivative operators on functions on the surface rather than as "physical" vectors. But everything works satisfactorily!) We are going to spend the rest of the time before the final project presentations studying a particularly interesting and historically significant example. Namely, let $\mathbb{H}$ as a set be

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

the "upper half-plane." We will think of this as an abstract surface *different from the plane or a part of the ordinary plane* by introducing the Riemannian metric with $E = G = \frac{1}{y^2}$ and $F = 0$, or

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{1}$$

We want to study the geometry on this abstract surface in detail and understand how it compares with other geometries we have studied in the course.

Questions

(A) To "warm up" and see what (1) does to make the geometry of this abstract surface significantly different from the usual geometry of the upper half-plane in $\mathbb{R}^2$, for each $u \in (0, 1)$ consider the curve $\alpha_u(t) = (0, 1) + t(0, -u) = (0, 1 - tu)$ for $t \in [0, 1]$. Using the metric from (1), compute the length of $\alpha_u(t)$ as a function of u. What happens in the limit as $u \to 1$, so $\alpha_u(1) \to (0, 0)$?

(B) Compute the Christoffel symbols and Gaussian curvature for this abstract surface. The result for the curvature should be surprisingly simple(!)

(C) Show that for any constant a, the curve $\beta(t) = (a, e^t)$ is a unit speed geodesic on this abstract surface.

(D) Show that if the constants a, b, c, d satisfy $ad - bc = 1$, then the following curve is a geodesic on this abstract surface[1]:

$$\gamma(t) = \left(\frac{ace^{2t} + bd}{c^2e^{2t} + d^2}, \frac{e^t}{c^2e^{2t} + d^2}\right) \tag{2}$$

[1] (continues at bottom of page 2) This is a pretty messy calculation by hand (although

(E) Show that $\gamma(t)$ from part (D) is an arc of a circle in the plane whose diameter is a segment along the x-axis. As $t \to \pm\infty$, show that the tangent vector to $\gamma(t)$ becomes vertical, so although $\gamma(t)$ never reaches the x-axis, it approaches the x-axis along a perpendicular direction as $t \to \pm\infty$.

(F) Show that given any two points P, Q in $\mathbb{H}$, either P, Q have the same x-coordinate so they are both on one of the geodesics from (C), or else there is a curve of the form (2) containing both P, Q (Hint: use the description given in part (E)!) Explain why the geodesic segment from P to Q can always be extended indefinitely in both directions.

(G) A *geodesic circle* in $\mathbb{H}$ with center at P and radius $r > 0$ should be the curve obtained by moving out from P a distance r along the geodesics through P in all possible directions. The distance should be measured as arc-length along the geodesic. Argue that geodesic circles exist with all centers and all radii (all values of $r > 0$).

(H) Right angles are defined using the metric to compute angles between vectors. In the tangent space, these are always right angles in the usual sense, so they are all the same. (Nothing to prove here, just notice the pattern.)

(I) Given a geodesic ℓ and a point P not on the geodesic, how many geodesics are there through P that do not meet ℓ?

(J) What's the geometry on this abstract surface $\mathbb{H}$ with the metric from (1)? Where have we seen it before?

Assignment

Group writeups due no later than Friday, May 6. (This will be the final assignment for the course apart from the final project presentations and papers.)

it's definitely do-able). If you want, you can use Maple to show that the differential equations for geodesics are satisfied. The "clean way" to prove this is to show that this curve is actually the image of the geodesic from part (C) under a certain transformation of $\mathbb{H}$ that preserves the metric (1). The cleanest way to describe that mapping is by writing (x, y) as the complex number $z = x + iy$. Then the mapping involved is the so-called *fractional linear* mapping $z \mapsto \frac{az+b}{cz+d}$.