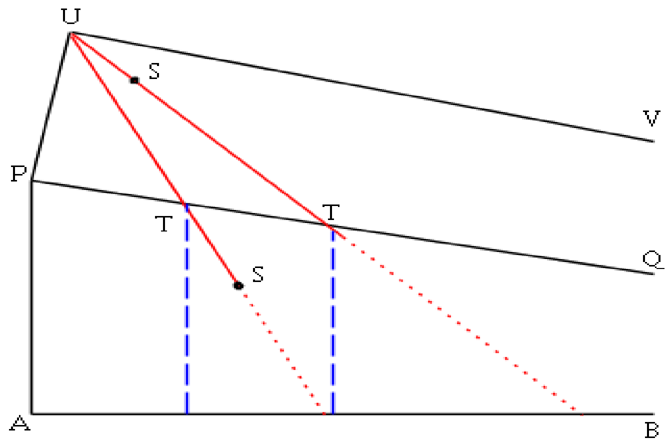

Figure 1: Figure for Propositition 3.

Proposition 3 *If $\overleftrightarrow{PQ}$ is parallel to $\overleftrightarrow{AB}$ in the direction of B and $\overleftrightarrow{UV}$ is parallel to $\overleftrightarrow{PQ}$ in the direction of Q (on the same side as B), then $\overleftrightarrow{UV}$ is parallel to $\overleftrightarrow{AB}$.*

(In other words, parallelism is transitive.) There are two cases, depending on whether $\overleftrightarrow{PQ}$ lies between the two other lines or outside the region between them. We will do the first of these; for the other, refer to McCleary's text.

- $PA \perp \overleftrightarrow{AB}$ and $UP \perp \overleftrightarrow{PQ}$ as in the Figure above.
- Say S is in $PQVU$. The goal is to show that $\overleftrightarrow{US}$ meets $\overleftrightarrow{AB}$.
- Since $\overleftrightarrow{UV}$ is parallel to $\overleftrightarrow{PQ}$ in the direction of Q, line $\overleftrightarrow{US}$ meets $\overleftrightarrow{PQ}$ at some point T. If we continue that line across $\overleftrightarrow{PQ}$, then by Proposition 1 from last time, the line must continue and meet $\overleftrightarrow{AB}$.
- On the other hand, suppose S is in $ABQP$.
- Postulate I says we can join US with a line segment crossing $\overleftrightarrow{PQ}$ at some T and Proposition 1 again shows that line must continue and cross $\overleftrightarrow{AB}$.

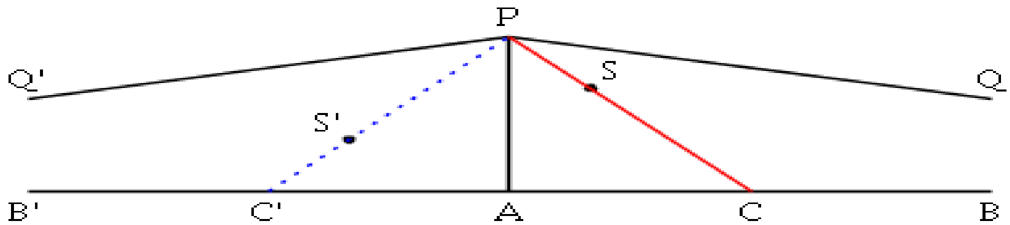

Figure 2: Figure for Proposition 4.

Proposition 4 *The angles of parallelism* $\Pi(AP)$ *in the direction of* B *and* $\Pi(AP)$ *in the direction of* B' *are equal if* B, B' *lie along the same line through* A*, but on opposite sides of* A*.*

- The proof consists of showing that the angles $\angle APS$ with S on the right side of PA yielding lines that meet $\overleftrightarrow{AB} = \overleftrightarrow{AB'}$ are exactly the same as the angles $\angle APS'$ with S' on the other side yielding lines that meet $\overleftrightarrow{AB}$.
- Suppose $\overrightarrow{PS}$ meets $\overrightarrow{AB}$ at C.
- Lay off a segment $AC' = AC$ on the other side of PA and join PC'.
- Then $\Delta PAC \cong \Delta PAC'$ (why?)
- This shows that $\angle S'PA = \angle SPA$ is the same angle on the other side and the line through S' meets $\overleftrightarrow{AB'}$ at C'.
- Hence the angles that "work" on the left contain all the angles that "work" on the right.
- Now, reverse the roles of the two sides and repeat the same argument.

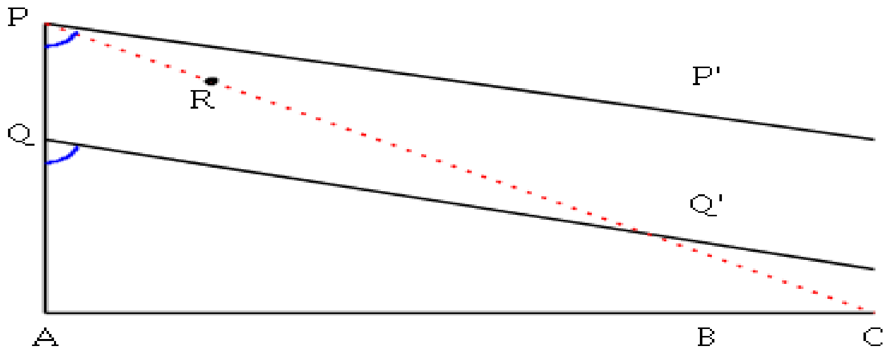

Figure 3: Figure for Proposition 6.

Proposition 5 *The angle of parallelism* $\Pi(AP)$ *depends only on the length of the segment* AP *and not on the exact locations of the points* A *or* P*.*

Proposition 6 *If Saccheri's HAA holds and* $AP > AQ$*, then* $\Pi(AP) < \Pi(AQ)$*. (That is, the angle of parallelism is monotone decreasing as a function of the length* AP*.*

- The proof of Proposition 6 consists of showing that both $\Pi(AP) = \Pi(AQ)$ and $\Pi(AP) > \Pi(AQ)$ lead to contradictions.

- If $\Pi(AP) = \Pi(AQ)$, then the result from Euclid I.27 shows that the parallels $\overrightarrow{QQ'}$ and $\overrightarrow{PP'}$ are themselves parallel. This means they have a common perpendicular line and that contradicts Theorem 3.14 in McCleary (one of Saccheri's results assuming HAA).

- If $\Pi(AP) > \Pi(AQ)$, then refer to the Figure above.

- There exists R in $ABP'P$ such that $\angle APR = \angle AQQ'$ (angles marked in blue in the Figure)

- But then $\overrightarrow{PR}$ must meet $\overleftrightarrow{AB}$ at some point because that line lies below the parallel $\overleftrightarrow{PP'}$, and it must also meet $\overleftrightarrow{QQ'}$.

- However the result of Euclid I.27 says that $\overleftrightarrow{PR}$ and $\overleftrightarrow{QQ'}$ cannot meet since they are parallel in Euclid's sense.

- This is a contradiction and it finishes the proof.