MATH 392 – Geometry Through History
Selected Solutions for Problem Set 3
March 1, 2016

I. Show under the assumption of HAA that given any angle $\angle ABC < 180°$, there is a line ℓ that is simultaneously parallel to $\overrightarrow{BA}$ in the direction of A and to $\overrightarrow{BC}$ in the direction of C. Hint: use point (3) on page 47 of McCleary.

Solution: First construct an angle bisector $\overrightarrow{BD}$ for the angle $\angle ABC$. By the assumption, both of the "halves," the angles $\angle ABD = \angle CBD = \theta$ are *acute*. (Recall that this construction uses only Postulates I - IV in Book I of Euclid, so it is valid in the hyperbolic case as well.) Now point (3) on page 47 of McCleary says that every acute angle is the angle of parallelism $\Pi(x)$ for some $x > 0$. Let D' be on the ray $\overrightarrow{BD}$ such that $BD' = x$ where $\Pi(x) = \theta$ for the θ above (one half the original angle). Then if we erect a perpendicular $\overrightarrow{D'E}$ in the direction of A to $\overrightarrow{BD}$, we will have that $\overrightarrow{D'E}$ and $\overrightarrow{BA}$ are parallel in the direction of A. Similarly, if we erect a perpendicular $\overrightarrow{D'F}$ in the direction of C to $\overrightarrow{BD}$, we will have that $\overrightarrow{D'F}$ and $\overrightarrow{BC}$ are parallel in the direction of C. Finally the points E, D', F must all be along a single line since $\angle ED'B = \angle FD'B$ are both right angles. The line ℓ is $\overleftrightarrow{EF}$.

II. Note: All of the following facts about 3-dimensional geometry are proved in McCleary. Suggestion: try to work out proofs for yourselves using just properties following from Euclid's Postulates I-IV (not V), then use the text to check your work.

(C) If T and T' are perpendicular planes, and ℓ is a line on T, then $\ell \perp T'$ is equivalent to $\ell \perp m$ for $m = T \cap T'$.

Solution: First note that the way this is stated means that we must prove two implications ("is equivalent to" means the same thing as "if and only if"). So to begin, assume that T and T' are perpendicular planes and ℓ is a line on T with $\ell \perp T'$. Suppose $\ell \cap T' = P$. The point P must be a point on the line $T \cap T' = m$. By definition (Definition 4.14 in McCleary), ℓ meets every line through P in T' at a right angle. Since m is one of those lines, $\ell \perp m$. Conversely, suppose ℓ is a line on T that meets $m = T \cap T'$ at a right angle. Since the planes T and T' are assumed perpendicular, there is a line ℓ' in T' that meets T in a right angle (see figure on right on page 59 in McClearly). Take any point R on ℓ' not lying in T and join it to $P = \ell \cap m$ with a line. That line $\overleftrightarrow{RP}$ also meets ℓ at a right angle by part (B). But then ℓ meets two different lines through P in T' at a right angle, so ℓ is perpendicular to T' by part (A).

(D) If $\ell_1 \perp T$ and $\ell_2 \perp T$, then ℓ_1, ℓ_2 are coplanar and non-intersecting (but not necessarily parallel – this is a difference in the hyperbolic case!)

Solution: We want to assume that ℓ_1 and ℓ_2 are *distinct* lines, of course, so they must meet T at two distinct points. Let $\ell_1 \cap T = P$ and $\ell_2 \cap T = Q$. Let T' be the plane containing Q and ℓ_1. Then T and T' are perpendicular planes by definition. We want to show that T'

must contain ℓ_2 as well. Note that $T' \cap T$ contains both points P and Q, so $T' \cap T = \overleftrightarrow{PQ}$. Let ℓ' be the line through Q perpendicular to $T' \cap T$ in T'. Then ℓ' is also perpendicular to T by part (C). But there is only one line through Q that is perpendicular to the plane T. Hence $\ell' = \ell_2$, and ℓ_2 must also lie in T'.

(E) Given a line ℓ and a plane T not containing ℓ, there exists a unique plane T' containing ℓ that is perpendicular to T.

Solution: Note that we really need to assume *in addition* that ℓ is not itself perpendicular to T (see the class notes where we discuss the analog of the Playfair Postulate for 3-dimensional hyperbolic geometry). If that is the case, since T does not contain ℓ, there is a point P on ℓ that is not in T. Let m be the line through P perpendicular to T. The plane containing ℓ and m contains ℓ and is perpendicular to T by definition. This shows existence. Any second such plane would have to contain ℓ and some line perpendicular to T passing through a point Q in ℓ. But then the two planes must coincide because of part (D) – any two lines distinct lines perpendicular to the same plane are coplanar.

IV. Why is the formula

$$\cosh(c/k) = \cosh(a/k)\cosh(b/k)$$

called the hyperbolic form of the Pythagorean theorem? Hint: Use III, part (C) and think about what happens as $k \to \infty$.

Solution: Part (C) of III should show that the Taylor series of $\cosh(t)$ at $t = 0$ looks like this:

$$\cosh(t) = \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots .$$

This series converges to $\cosh(t)$ for all real t (for the same reason that the Taylor series for e^t converges for all t). If we use this on the given equality we get

$$1 + \frac{(c/k)^2}{2!} + \frac{(c/k)^4}{4!} + \cdots = (1 + \frac{(a/k)^2}{2!} + \frac{(a/k)^4}{4!} + \cdots)(1 + \frac{(b/k)^2}{2!} + \frac{(b/k)^4}{4!} + \cdots).$$

Subtracting 1 from both sides, multiplying out the right side and rearranging a bit yields

$$\frac{c^2}{2k^2} = \frac{a^2}{2k^2} + \frac{b^2}{2k^2} + \frac{1}{k^4}(\text{other terms}).$$

Hence if we multiply both sides by $2k^2$ and let $k \to \infty$ we get

$$c^2 = a^2 + b^2.$$

Recall that a, b, c were the lengths of the sides in a (hyperbolic) right triangle. Hence if $k \to \infty$, or equivalently if $a, b, c \to 0$, in the limit, we have the Euclidean Pythagorean relation.