MATH 392 – Geometry Through History
Solutions for Problem Set 2
Due: Friday, February 12

I. Proposition 30 in Book I of Euclid's *Elements* says: *Two lines that are parallel to the same line are parallel to each other.*

Here's the proof Euclid provides: Let it be given that the lines $\overleftrightarrow{AB}$ and and $\overleftrightarrow{EF}$ are parallel and the lines $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ are parallel. Pick any two points G on the line $\overleftrightarrow{AB}$ and K on the line $\overleftrightarrow{CD}$, draw the line segment GK (and extend it – Postulates I and II). Let H be the intersection of the line $\overleftrightarrow{GK}$ with the line $\overleftrightarrow{EF}$. Then $\angle AGH = \angle FHG$ and $\angle FHG = \angle DKG$ (Proposition 29). Hence also $\angle AGH = \angle DKG$ (Common Notion 1). Therefore, $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ (Proposition 27).

(A) There is a point here that is questionable in the sense that it does not really follow from any of the Postulates or the previous results in Book I. What is this issue?

Solution: The issue is that Euclid states neither an axiom guaranteeing, nor a proof for the statement that, the point H here must exist or that it can actually be constructed.

(B) To fix the problem you noted in part (A), prove the following statement: *If a line intersects one of a pair of parallel lines, then it intersects the other one as well.* Hint: You should take "line" here to mean a line extended as far as possible in both directions (Postulate 2). Argue by contradiction and note that two lines that do not intersect are parallel by definition. Your proof *should* make use of results that depend on Postulate V.

Solution: Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be the parallel lines, suppose that G is a point on $\overleftrightarrow{AB}$, the line $\overleftrightarrow{GH}$ intersects $\overleftrightarrow{AB}$ at G, but $\overleftrightarrow{GH}$ is not equal to the line $\overleftrightarrow{AB}$. We claim that $\overleftrightarrow{GH}$ must also meet $\overleftrightarrow{CD}$. Arguing by contradiction, suppose it does not meet $\overleftrightarrow{CD}$. Then by definition $\overleftrightarrow{GH}$ and $\overleftrightarrow{CD}$ are parallel. However, we know that the statement in the Playfair Postulate follows from Postulates I-V. Namely given a line, like $\overleftrightarrow{CD}$, and a point not on that line, like G, there exists *exactly one parallel* to $\overleftrightarrow{CD}$ passing through G. Since we now know that $\overleftrightarrow{AB}$ and $\overleftrightarrow{GH}$ are both parallel to $\overleftrightarrow{CD}$ and contain G, this implies that they must be the same line. But that contradicts our assumption that $\overleftrightarrow{AB}$ and $\overleftrightarrow{GH}$ were different lines.

II. Give a proof of Proposition 35 in Book I of the *Elements* in the cases that the sides CD and EF of the two parallelograms overlap. (See the slides on Book I.)

Solution: Let $ABDC$ and $ABFE$ be the parallelograms sharing the base AB and such that the sides CD and EF lie along the same line parallel to the line containing AB. The case not considered in class (or in Euclid) is the one where the point F lies between C and D, so the top edges of the parallelgrams overlap. First note that the lengths satisfy $EF = EC + CF$ and $CD = CF + FD$ in this case. In addition $EF = CD = AB$ by Proposition I.34. Hence by Common Notion 3,

$EC = FD$. Since $\overleftrightarrow{AE}$ and $\overleftrightarrow{BF}$ are parallel and $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are parallel (Proposition I.33), it follows that $\angle AEC = \angle BFD$ and $\angle ACE = \angle BDF$ (Proposition I.29). Hence $\Delta AEC \cong \Delta BFD$ by Proposition I.26 (the AAS congruence). (There are, of course, several other ways to deduce this using both SAS and SSS.) It follows that those two triangles have the same area. Now the parallelograms $ABDC$ and $ABFE$ consist of two congruent triangles, plus the shared quadrilateral $ABFC$. Hence they must have the same area (Common Notion 2), and that is what we wanted to show.

III. Assume Postulates I - IV of the *Elements* hold. Consider the following statements:

(A) Euclid's Postulate V.

(B) A line perpendicular to one ray of an acute angle intersects the other ray as well.

(C) Through any point in the interior of an angle less than a straight angle (180°), there passes a line meeting each of the two rays at points other than the vertex.

(D) The sum of the angles in any triangle is 180°.

Show that all these statements are equivalent by showing that (A) $\Rightarrow$ (B), (B) $\Rightarrow$ (C), (C) $\Rightarrow$ (D). Hints: We know from class that (D) $\Rightarrow$ (A). So once you show those three implications, all four statements are equivalent. Also, see Problem 3.5 in McCleary, which gives an "attempted proof" of Postulate V by A.-M. Legendre (1804). You *can* use this proof under the assumption that (C) is true. Do you see why? And do you see why Legendre's proof fails if we don't know (C) is true?

Solutions:

(A) $\Rightarrow$ (B): Call the acute angle $\angle BAC$ and suppose the perpendicular $\overleftrightarrow{DC}$ meets the ray $\overrightarrow{AC}$ at the point C (taking D to be a point in the interior of the angle). The two lines $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$ have the transversal line $\overleftrightarrow{AC}$ and the sum of the (interior) angles on the side toward the angle is $\angle BAC + \angle ACD < 180^\circ$ (since $\angle BAC$ is acute and $\angle ACD = 90^\circ$). Hence Postulate V implies immediately that $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ meet on that side of the transversal line $\overleftrightarrow{AC}$.

(B) $\Rightarrow$ (C): Given the angle $\angle ABC$, construct $\overrightarrow{BD}$ bisecting the angle (Proposition I.9). Let P be any point in the interior of the angle $\angle ABC$ and construct a line $\overleftrightarrow{PQ}$ through P meeting $\overrightarrow{BD}$ in a right angle (use Proposition I.11 if P is on the bisecting ray and I.12 otherwise). Since we assumed $\angle ABC$ was less than 180°, both of the angles $\angle ABD$ and $\angle DBC$ are acute. Moreover the line $\overleftrightarrow{PQ}$ crosses one ray of each of those angles. Therefore, it crosses the other rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ at points different from B by (B).

(C) $\Rightarrow$ (D): Refer to the diagram in Problem 3.5 in McCleary. The triangle ΔABC is given and we want to show that the angle sum is 180° exactly. By the Saccheri-Legendre theorem, we know that angle sum is $180^\circ - \delta$ for some $\delta \geq 0$. So aiming for a contradiction, assume $\delta > 0$. Construct $\Delta BCD \cong \Delta ABC$. Since D is in the interior of the angle $\angle BAC$, part (C) implies there is a line through D that meets $\overrightarrow{AB}$ at E and $\overrightarrow{AC}$ at F. Now consider the large triangle ΔAEF, made up of the four smaller triangles ΔABC, ΔBED, ΔBCD, and ΔCDF. Add up all of the angles in those four triangles in two ways. First, separate out the angles from the large triangle ΔAEF and notice that the other angles form three groups each of which adds to 180°. The total is

$$\angle EAF + \angle AFE + \angle FEA + 540°. \tag{1}$$

On the other hand, if we add the angles one triangle at a time the total can be seen to be equal to

$$(180° - \delta) + (180° - \delta) + (180° - \epsilon_1) + (180° - \epsilon_2). \tag{2}$$

for some $\delta, \epsilon_1, \epsilon_2 \geq 0$. The first two parentheses come from ΔABC and ΔBCD and the others come from the remaining two triangles. We have used the Saccheri-Legendre theorem for them. Comparing (1) and (2), we see that

$$\angle EAF + \angle AFE + \angle FEA = 180° - 2\delta - \epsilon_1 - \epsilon_2 \leq 180° - 2\delta.$$

Note the 2δ. This says that if we repeat the whole construction n times with the larger and larger triangles we are considering then after n steps we will have a triangle whose angle sum is $\leq 180° - 2^n\delta$. If $\delta > 0$, then this will be negative for n sufficiently large, and that is a contradiction. Hence the only possible value is $\delta = 0$ and that is what we wanted to show.

The interesting thing here historically is that Legendre essentially assumed the statement in (C) was "obviously true" and used it without giving a justification. This problem shows that the innocent-looking statement (C) is actually equivalent to Postulate V(!)