MATH 392 – Geometry Through History
Solutions for Problem Set 1
Friday, February 5

2.8.

(a) Proposition I.9: With straightedge and compass, one can bisect a given angle. Proof: Say the angle is $\angle ABC$. Pick any one point P on the ray BA, and lay off a segment of length equal to BA on the other ray BC (Proposition I.3). Call the end point of that segment Q, and join PQ (Postulate I). Construct the equilateral triangle ΔPQR, with R on the opposite side of PQ from B (Proposition I.1) and join BR. We claim that BR bisects $\langle ABC$. To prove this, note that in ΔPBQ, we have $\angle BPQ = \angle BQP$ since $BP = BQ$ (Proposition I.5). For a similar reason, in the equilateral triangle ΔPQR, $\angle QPR = \angle PQR$ (Proposition I.5). Hence by Common Notion 2,

$$\angle BPR = \angle BPQ + \angle QPR = \angle BQP + \angle PQR = \angle BQR.$$

In addition, $BP = BQ$ by construction and $PR = QR$ since ΔPQR is equilateral. Hence by SAS (Propositon I.4), $\Delta BPR \cong \Delta BQR$. It follows that the corresponding angles $\angle PBR$ and $\angle QBR$ are equal, so the segment BR bisects the angle $\angle PBQ$.

(b) Proposition I.10: One can bisect a given segment. Proof: Let AB be the line segment. Construct the equilateral triangle ΔABC (Proposition I.1). Construct CD bisecting $\angle ACB$ (Proposition I.9) and extend to E lying on the segment AB. (Note: That finding such a point E is possible really doesn't follow from any of Euclid's axioms; for this you need something like *Pasch's Axiom* on page 17 of McCleary. We claim $AE = EB$, so we have bisected the segment AB. To prove this, consider the triangles ΔACE and ΔBCE. First, note that $CA = CB$ by the fact that ΔABC is equilateral. Then $\angle ACE = \angle BCE$ since CE is constructed to bisect $\angle ACB$. Finally EC is common to the two triangles. Hence $\Delta ACE \cong \Delta BCE$ by SAS (Proposition I.4). Therefore the corresponding sides AE and BE are equal, so E bisects AB.

(c) Proposition I.11: One can "erect" a perpendicular to a given line from any point on that line. Proof: Call the point E and take A to be any point on the line other than E. Lay off a line segment $EB = EA$ on the other side of E along the line (Proposition I.3). Then construct the equilateral triangle ΔABC (Proposition I.1) and join CE (Postulate 1). We claim $\angle AEC$ and $\angle BEC$ are right angles so EC is the required perpendicular. This follows from the same proof given for Proposition I.10 in part (b). As there $\Delta ACE \cong \Delta BCE$ by Proposition I.8 (SSS) Hence the corresponding angles $\angle AEC$ and $\angle BEC$ are equal. But also $\angle AEC + \angle BEC = 2$ right angles so each of $\angle AEC$ and $\angle BEC$ is a right angle (Common Notion 3).

(d) Proposition I.18: In any triangle the greater side is opposite the greater angle. Proof: That is, we have to show that if we have a triangle ΔABC and (for instance) $BC > AC$, then $\angle BAC > \angle ABC$. Since $AC < BC$ we can lay off a segment of length AC starting at C and ending at a point D between B and C (Proposition I.3). Join AD (Postulate I). The triangle

ΔADC is isosceles, so $\angle CDA = \angle CAD$. The angle $\angle ADC$ is an exterior angle to the triangle ΔABD. Hence by Proposition I.16, $\angle CDA > \angle ABD$. But also by Common Notion 5, we have $\angle BAC > \angle CAD$. Putting these together we get $\angle BAC > \angle CAD = \angle CDA > \angle ABD$. This is what we wanted to show.

2.9. Say AB and CD are two lines with transversal line AC and $\angle BAC + \angle DCA < 180°$ (i.e. less than two right angles). We need to show that AB and CD, if extended indefinitely will meet on the side containing B and D. If not, then those two lines would be parallel (they cannot meet on the other side of the transversal – the sum of the angles on that side is greater than two right angles so the lines must diverge on that side by Proposition I.17). However, then Proposition I.29 implies that $\angle DAC = \angle ECA$ (for any point E on the other side of C on the line CD. But then Proposition I.13 implies $\angle DCA = 180° - \angle ECA = 180° - \angle BAC$, so $\angle BAC + \angle DCA = 180°$, which contradicts the hypothesis.

2.13. Let ΔABC be inscribed in a circle with AB a diameter and C on a semicircle cut out by the diameter. We claim $\angle ACB$ is a right angle, *assuming Postulate V*. Let O be the center of the circle which is a point on AB and join OC (Postulate 1). Then ΔAOC and ΔBOC are isosceles triangles since $OA = OC = OB$ (definition of a circle). By Proposition I.5, we have $\angle OAC = \angle OCA$ and $\angle OCB = \angle OBC$. By Proposition I.32 (which depends on Postulate V), the sum of the angles in ΔABC is equal to two right angles. But that sum equals $2(\angle OCA + \angle OCB) = 2\angle ACB$. Hence $\angle ACB$ is a right angle (Common Notion 3).

2.14. There is a solution in the back of McCleary's book that uses several different sorts of constructions based on Euclid's Proposition I.43 that we did not discuss. It has the merit of producing what are probably the *most efficient* constructions. But there is also another way that is based entirely on the fact proved in 2.13. Here's an outline of this alternative method:

Step 0: First, note that given segments of lengths $x > y$, it is easy to construct segments of lengths $x + y$ and $x - y$ by laying off segments along lines using Proposition I.3. Then note that if we have any triangle ΔABC with AB a diameter of the circumscribed circle, then $\angle ACB$ is a right angle. Hence if we drop the perpendicular CD from C to AB (using Proposition I.12) then ΔABC, ΔACD and ΔBCD are all *similar triangles.*

Step 1: given a segment AD of length x and a segment DB of length 1, make the segment AB of length $1 + x$, construct the circle with that diameter, and erect the perpendicular to AB from the point D, call (one of) the intersection points with the circle C and consider the triangle ΔABC and the two smaller triangles ΔACD and ΔBCD. These are similar by the observation above. If z is the length of CD, then by the fact that corresponding parts of similar triangles are in the same proportion:

$$\frac{x}{z} = \frac{z}{1}, \text{ or } x = z^2.$$

This can be read two ways:

- If we know x, it gives a construction of $z = \sqrt{x}$.

- On the other hand if we know $z = DC$, we can start from $BD = 1$, do the construction of the perpendicular to the diameter, lay off DC on that line, construct the perpendicular line to BC, find the point A where that intersects the BD extended, and call $AD = x$. This gives a construction of z^2 if we know z.

Step 2: Now suppose segments of lengths $x > y$ are known. Consider this construction:

- Construct segments of lengths $x + y$ and $x - y$ using Step 0
- Construct segments of lengths $(x + y)^2$ and $(x - y)^2$ using Step 1,
- Construct a segment of length

  $$(x + y)^2 - (x - y)^2 = 4xy$$

  by Step 0 again
- Construction a segment of length xy by bisecting the previous segment twice (Proposition I.10). Putting all of this together gives a construction of a segment of length xy.

The only remaining part is to construct a segment of length $\frac{1}{x}$ given a segment of length x. To do this proceed as in the second part of Step 1, except make $z = 1$ and x the known segment along the diameter of the circle. By similar triangles, if y is the remaining part of the diameter, we will have

$$\frac{y}{1} = \frac{1}{x}$$

so y is the reciprocal of x.