MATH 392 – Geometry Through History
Solutions/Lecture Notes for Class Monday, February 8

Background

Recall that on Friday we had started into a rather complicated proof of a result showing that the usual fact about the sum of the angles in a triangle cannot be assumed if we want to try to prove Postulate V from the other postulates. In fact, Postulates I-IV plus that statement is equivalent to Postulates I-V as Euclid stated them. This proof is attributed to the medieval Islamic mathematician Nasir al-Din al-Tusi (although this has sometimes been questioned).

Theorem 1 *Assume that Euclid's Postulates I-IV (and all the additional facts such as Pasch's Axiom and the Axiom of Continuity that Euclid did not state explicitly, but that are needed for complete proofs of Propositions 1 - 28) hold. Assume in addition that the angle sum in every triangle is two right angles (*$180°$*). Then the statement of Postulate V also holds (as a theorem).*

Questions

We said to start that, given a transversal line falling on two lines making angles on one side summing to less than $180°$, (if necessary) we could replace that transversal with a different transversal for which one of the angles was a right angle and for which the angle sum on that one side did not change. (Note that showing the two lines meet using this other transversal is sufficient for what we are trying to show!)

I. Prove that we can always construct this other transversal. (Hint: Drop a perpendicular. You will need to use the assumption that the angle sum in all triangles is $180°$*.)*

Solution: Say the lines are AB and CD where AC is the transversal line and $\angle BAC + \angle DCA < 180°$. Drop a perpendicular AK to CD (where K is the foot, a point lying on the line CD). In the triangle ΔAKC, note that $\angle KAC + \angle ACK = 90°$ under the assumption that all triangles have angle sum equal to $180°$. Using the line AK as the new transversal, we have angle

$\angle BAK$ and the right angle $\angle AKD$. But

$$\angle BAK + 90^\circ = \angle BAK + \angle KAC + \angle ACK = \angle BAC + \angle ACD$$

so the sum of the angles for the new transversal is the same the angle sum for the old transversal on that side. (This leads to a diagram like the one we were considering in class last Friday.)

It will help to refer to the diagram we drew on the board, or to page 32 in McCleary's book.

Now, assuming the transversal line AC makes a right angle with one line CD and an acute angle with the other line AB, we began the argument as follows: Let G_1 be an arbitrary point on the line AB on the side of the transversal with the angle sum less than 180°. Drop a perpendicular to the transversal AC from G_1 and call the foot H_1. If $AH_1 > AC$, then the line CD enters the triangle ΔAH_1G_1 along the side AH_1. The line CD is parallel to H_1G_1 since both make right angles with AH_1 (which is AC, extended). Hence Pasch's Axiom implies CD must exit the triangle ΔAH_1G_1 through the other side – AG_1 = the extension of AB – and we are done in this case.

II. Why can't we just stop there? Why do we need to do the next part of the argument?

Solution: We cannot just stop there since there is nothing in Postulates I - IV (or Pasch's Axiom for that matter) that implies directly that the point G_1 can be chosen so that $AH_1 > AC$. In other words, the next steps are necessary.

If $AH_1 \leq AC$, then we argued as follows. By the Axiom of Continuity, (also called the *Archimedean Axiom* – see McCleary, p. 17), using Euclid's Proposition I.3, we can lay off enough equal segments

$$AH_1 = H_1H_2 = \cdots = H_{n-1}H_n$$

to make H_n lie "strictly past" C along the line AC (extended using Postulate II). We can also lay off equal segments

$$AG_1 = G_1G_2 = \cdots = G_{n-1}G_n$$

(with the same number n) along the line AB (extending it as needed using Postulate II). The theorem will be proved if we can show that for all $i \geq 2$, the point H_i is the foot of the perpendicular dropped from G_i to the line AC, extended to AH_n. (Reason: We will have CD entering one side of the triangle ΔAH_nG_n and we can argue as before, using Pasch's Axiom, that the line CD, extended using Postulate II, must exit that triangle along the side AG_n, which is the extension of the line AB.)

So we need to show that G_iH_i is perpendicular to AH_n for all $i = 2, \ldots, n$. To start, suppose K is the foot of the perpendicular from G_2 to AH_n. We must show $K = H_2$.

III. Construct AL perpendicular to AH_1 with $AL = H_1G_1$. Show that $\angle G_1AL = \angle AG_1H_1$ (using the assumption about the angle sum in triangles). Conclude that $\Delta G_1H_1A \cong \Delta G_1LA$, hence $\angle G_1LA$ is a right angle.

Solution: Extending AC beyond the line AL to a point N, we have $\angle NAL = 90^\circ$ and then

$$180^\circ = 90^\circ + \angle LAG_1 + \angle G_1AH_1.$$

On the other hand, by the assumption about the angle sum in the triangle ΔG_1H_1A, we also have

$$180^\circ = \angle G_1H_1A + \angle G_1AH_1 + \angle H_1G_1A = 90^\circ + \angle G_1AH_1 + \angle H_1G_1A.$$

Comparing the last two equations yields $\angle LAG_1 = \angle H_1G_1A$. Now we also have $AL = G_1H_1$ by construction and the side AG_1 is shared by the two triangles ΔG_1H_1A and ΔG_1LA. By SAS (Proposition I.4), the triangles are congruent. It follows that the corresponding angles $\angle G_1H_1A$ and $\angle G_1LA$ are congruent, so they are both right angles.

Now construct a point M on the line segment KG_2 so that $KM = H_1G_1$.

IV. Show that $\angle H_1G_1K = \angle MKG_1$ (again use the assumption about angle sums in a triangle). Deduce that $\Delta MG_1K \cong \Delta H_1KG_1$ and $\angle KMG_1$ is a right angle.

Solution: This is similar to step III above: At K we have a straight angle along the line AC, so

$$180^\circ = 90^\circ + \angle MKG_1 + \angle G_1KH_1$$

But in the triangle ΔH_1KG_1 we have

$$180^\circ = 90^\circ + \angle H_1G_1K + \angle G_1KH_1$$

Combining these two equations, we get $\angle MKG_1 = \angle H_1G_1K$. It follows by SAS that $\Delta MG_1K \cong \Delta H_1KG_1$ since $KM = H_1G_1$ by construction and the side G_1K is common to the triangles. Then, the corresponding angles $\angle KH_1G_1$ and $\angle KMG_1$ are equal, so $\angle KMG_1$ is a right angle.

V. Explain why M, G_1, and L must be collinear. (Note: that was not assumed, but it follows from what we have done to this point.)

Solution: This follows because the sum

$$\angle MG_1K + \angle KG_1H_1 + \angle H_1G_1A + \angle AG_1L$$

is $180°$. The first two add to $90°$ because they equal the other two angles in one of the congruent right triangles ΔKMG_1 or ΔKH_1G_1; and the other two also add to $90°$ for the same sort of reason using the congruent right triangles ΔG_1H_1A and ΔG_1LA.

VI. Next, show that $\Delta MG_2G_1 \cong \Delta LG_1A$. Which congruence criterion are you using? Be sure it depends only on Postulates I - IV and Propositions 1 - 28.

Solution: Since $\angle KMG_1$ is a right angle, so is $\angle G_1MG_2$. This is the same as $\angle G_1LA$ by step III above. Moreover, $\angle G_2G_1M = \angle AG_1L$ since those are two vertical angles at the intersection of two lines as in Proposition I.15 (A, G_1, G_2 lie along one line, and L, G_1, M lie along another by step V). Finally $AG_1 = G_1G_2$ by construction. Hence $\Delta MG_2G_1 \cong \Delta LG_1A$ by ASA (Proposition I.26 – note that that proposition comes before I.29, so it depends only on Postulates I - IV(!))

VII. Now deduce that $H_1K = AH_1$ which implies that $K = H_2$ from the construction of the points H_i.

Solution: Using the triangle congruences from Steps IV, VI, III (in that order) we have

$$H_1K = MG_1 = LG_1 = AH_1.$$

Since point H_2 was constructed to make $H_1H_2 = AH_1$, and K, H_2 both lie along the line AH_n, we get $K = H_2$.

VIII. What technique of proof would be most efficient to continue and show H_i is the foot of the perpendicular from G_i to AH_n for all i? Can you see how that would go, without writing out all of the details?

Solution: This would be a great candidate for a proof by mathematical induction. What we did above is essentially the base case for the induction. The induction step would be to prove that H_k is the foot of the perpendicular to AH_n through G_k, under the assumption that H_{k-1} is the foot of the perpendicular to AH_n through G_{k-1}. The proof would be similar, but it would require considering more triangles.