MATH 392 – Geometry Through History
Class Monday, February 8

Background

Recall that on Friday we had started into a rather complicated proof of a result showing that the usual fact about the sum of the angles in a triangle cannot be assumed if we want to try to prove Postulate V from the other postulates. In fact, Postulates I-IV plus that statement is equivalent to Postulates I-V as Euclid stated them. This proof is attributed to the medieval Islamic mathematician Nasir al-Din al-Tusi (although this has sometimes been questioned).

Theorem 1 *Assume that Euclid's Postulates I-IV (* and *all the additional facts such as Pasch's Axiom and the Axiom of Continuity that Euclid did not state explicitly, but that are needed for complete proofs of Propositions 1 - 28) hold. Assume in addition that the angle sum in every triangle is two right angles (*$180°$*). Then the statement of Postulate V also holds (as a theorem).*

What we will do today is work out the remainder of this proof in our discussion groups.

Questions

We said to start that, given a transversal line falling on two lines making angles on one side summing to less than $180°$, (if necessary) we could replace that transversal with a different transversal for which one of the angles was a right angle and for which the angle sum on that one side did not change. (Note that showing the two lines meet using this other transversal is sufficient for what we are trying to show!)

I. Prove that we can always construct this other transversal. (Hint: Drop a perpendicular. You will need to use the assumption that the angle sum in all triangles is $180°$*.)*

Now, assuming the transversal line AC makes a right angle with one line CD and an acute angle with the other line AB, we began the argument as follows: Let G_1 be an arbitrary point on the line AB on the side of the transversal with the angle sum less than $180°$. Drop a perpendicular to the transversal

AC from G_1 and call the foot H_1. If $AH_1 > AC$, then the line CD enters the triangle ΔAH_1G_1 along the side AH_1. The line CD is parallel to H_1G_1 since both make right angles with AH_1 (which is AC, extended). Hence Pasch's Axiom implies CD must exit the triangle ΔAH_1G_1 through the other side and we are done in this case.

II. Why can't we just stop there? Why do we need to do the next part of the argument?

If $AH_1 \leq AC$, then we argued as follows. (By the Axiom of Continuity, also called the *Archimedean Axiom* – see McCleary, p. 17), using Euclid's Proposition I.3, we can lay off enough equal segments

$$AH_1 = H_1H_2 = \cdots = H_{n-1}H_n$$

to make H_n lie "strictly past" C along the line AC (extended using Postulate II). We can also lay off equal segments

$$AG_1 = G_1G_2 = \cdots = G_{n-1}G_n$$

(the same number n) along the line AB (extending it as needed using Postulate II). The theorem will be proved if we can show that for all $i \geq 2$, the point H_i is the foot of the perpendicular dropped from G_i to the line AC, extended to AH_n. (Reason: We will have CD entering one side of the triangle ΔAH_nG_n and we can argue as before, using Pasch's Axiom, that the line CD, extended using Postulate II, must exit that triangle along the side AG_n, which is the extension of the line AB.)

So we need to show that G_iH_i is perpendicular to AH_n for all $i = 2, \ldots, n$. To start, suppose K is the foot of the perpendicular from G_2 to AH_n. We must show $K = H_2$.

III. Construct AL perpendicular to AH_n with $AL = H_1G_1$. Show that $\angle G_1AL = \angle AG_1H_1$ (using the assumption about the angle sum in triangles). Conclude that $\Delta G_1H_1A \cong \Delta G_1LA$, hence $\angle G_1LA$ is a right angle.

Now construct a point M on the line segment KG_2 so that $KM = H_1G_1$.

IV. Show that $\angle H_1G_1K = \angle MKG_1$ (again use the assumption about angle sums in a triangle). Deduce that $\Delta MG_1K \cong H_1KG_1$ and $\angle KMG_1$ is a right angle.

V. Explain why M, G_1, and L must be collinear. (Note: that was not assumed, but it follows from what we have done to this point.)

VI. Next, show that $\Delta MG_2G_1 \cong \Delta LG_1A$. Which congruence criterion are you using? Be sure it depends only on Postulates I - IV and Propositions 1 - 28.

VII. Now deduce that $H_1K = AH_1$ which implies that $K = H_2$ from the construction of the points H_i.

VIII. What technique of proof would be most efficient to continue and show H_i is the foot of the perpendicular from G_i to AH_n for all i? Can you see how that would go, without writing out all of the details?