MATH 392 – Geometry Through History
Review Sheet for Midterm Exam
March 3, 2016

General Information

As announced in the course syllabus and the online schedule, the midterm exam will be given at the end of the first week after spring break. If we decide to do the exam in class, it will be Friday, March 18. I'm open to an evening time too, though, in case there is sentiment that a format allowing you more time to work would be preferable. For instance, it would be possible to do the exam on Thursday evening, March 17, say 7-9pm. If we decided to do that, the normal Friday class would be cancelled. We can discuss this in class on Friday, March 4. (In any case, there will be no other written assignments due between now and the exam. So you should have ample time to prepare either way.)

Format

The exam will cover the material on Euclidean and hyperbolic geometry that we have studied since the start of the semester (i.e. Problem Sets 1 - 3 and the first quiz). There will be roughly equal coverage of the Euclidean and hyperbolic cases.

You should know the statements of Postulates I - V from Euclid, Pasch's Axiom, and definitions of terms we have used (especially the *revised* notion of parallelism and the angle of parallelism for the hyperbolic case, pencils of parallel lines, horocycles, horospheres.)

Know the statement of the Bolyai-Lobachevsky theorem, and be prepared to use it to deduce facts about angles of parallelism in the hyperbolic case.

I will ask you to construct one or two short proofs you have not seen before (comparable to the proof from Quiz 1; also see the Review/Practice Problems below). Other questions will ask for one or two of the more substantial assigned proofs listed below.

I might also include a short answer question with several parts on aspects of the historical development, so you should know approximately when each of the following key figures lived, what their main contributions were, and how they figure into what we have done so far:

- Euclid of Alexandria
- Proclus
- Girolamo Saccheri, S.J.
- Adrien-Marie Legendre
- Janos Bolyai
- Carl Friedrich Gauss
- Nikolai Lobachevsky

Assigned Proofs

1. The proof of Proposition 29 in Book I of Euclid's *Elements*. Also – what is especially notable about this Proposition?

2. The proof of Proposition 38 in Book I of the *Elements*, assuming Proposition 37.

3. The proof of the Saccheri-Legendre Theorem (Theorem 2.2 in McCleary). Also know that this result holds in both Euclidean and hyperbolic geometry. Why?

4. The proof that if $ABCD$ is a Saccheri quadrilateral on the base AB, then

   - if HAA holds, then $AB < CD$ and the angle sum in any triangle is less than 2 right angles,
   - if HOA holds, then $AB > CD$ and the angle sum in any triangle is greater than 2 right angles.

   (This is covered in Theorem 3.12 in McCleary.)

5. The proof that the ratio of lengths of concentric arcs of two horocycles intercepted by two lines in the parallel pencil is an exponential function of the distance between the arcs. (This is Theorem 4.9 in McCleary; we did the same proof in class as well.)

Review/Practice Problems

(A) Assuming Postulate V, prove that the three internal angle bisectors of a triangle meet in a single point. (Where does your proof use Postulate V?) Hint: Take two bisectors that meet and join the intersection point to the third vertex. Under what conditions does that line bisect the angle at that third vertex?

(B) Assuming Postulate V, prove that the three perpendicular bisectors of the sides of a triangle meet in a single point. (Where did you use Postulate V?) Hint: You can use the same strategy as in (A).

(C) Look at the figure in problem 4.20 on page 70 in McCleary. Show that in the hyperbolic plane, it is possible to construct three nonintersecting hyperbolic lines as in the top left diagram – that is show how to construct three lines $\ell_1, \ell_2,$ and ℓ_3 and points A, A' on ℓ_1, B, B' on ℓ_2 and C, C' on ℓ_3 such that $\overrightarrow{AA'}$ is parallel to $\overrightarrow{BB'}$ in the direction of A' and B', and (at the same time) $\overrightarrow{B'B}$ is parallel to $\overrightarrow{CC'}$ in the direction of B and C', and (at the same time) $\overrightarrow{A'A}$ is parallel to $\overrightarrow{C'C}$ in the direction of A and C. Then generalize your argument to show that there are similar "n-gons" of parallel lines for any $n \geq 3$.

(D) Determine the derivative $\frac{d}{dx}(\Pi(x))$ and show that it is always negative.

(E) Let $x, y > 0$. Express the angle of parallelism $\Pi(x + y)$ in terms of $\Pi(x)$ and $\Pi(y)$.