MATH 392 – Geometry Through History
Final Class Before Project Presentations
April 29, 2016

As you have now seen, the abstract surface $\mathbb{H}$ consisting of the upper half-plane with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

is what is called a "model" of the geometry of the hyperbolic plane we studied earlier in the semester(!) In other words, it is a surface constructed using the tools from differential geometry whose geodesics have the same properties as the lines in hyperbolic plane. The usual name for this surface is the *Poincaré upper half-plane model.* The existence of examples like this one in effect provides additional evidence for the validity of hyperbolic geometry.

As we said earlier in the semester, non-Euclidean geometries first appeared in published form in work of Lobachevsky (1830) and Bolyai (1832). Their work was to a great extent anticipated by unpublished work of Gauss done in the early 1800's. As their papers became better known and accepted by other mathematicians, there were still (at least in some quarters) what I will call "lingering doubts" about whether hyperbolic geometry was really a consistent mathematical system. Bolyai and Lobachevsky certainly worked under the understanding that their non-Euclidean geometries were just as consistent as the Euclidean geometry with Postulate V. However, they did not provide any *proofs* of such statements; they merely developed all the properties of the hyperbolic plane and exhibited all the ways hyperbolic geometry differed from Euclidean geometry.

The first mathematician who really addressed the question of proving consistency was the Italian geometer Eugenio Beltrami (1835-1900). What he did was to use the differential geometry of surfaces developed by Gauss and others to produce an explicit surface whose intrinsic geometry would be the same as (a portion of) the hyperbolic plane. This was even a *surface of revolution*, obtained by rotating the curve called the unit speed curve called the *tractrix*

$$\alpha(s) = \left(e^{-s}, \ln(e^s + \sqrt{e^{2s} - 1}) - \frac{\sqrt{e^{2s} - 1}}{e^s}\right)$$

about the x-axis. It is a nice (but somewhat involved) exercise to verify that the resulting surface, often called the *pseudosphere* (see Figure 1) has constant Gaussian curvature $K = -1$ (the same as our abstract surface $\mathbb{H}$). Beltrami then studied the geodesics on that surface and showed that geodesic triangles have the form we saw in hyperbolic geometry (the angle sums are less than 180°, etc.)

Now the key point here is that this surface with its non-Euclidean intrinsic geometry has been constructed *inside of the familiar Euclidean* $\mathbb{R}^3$. If there were any contradiction involved in its properties, they would also

Figure 1: Plaster model of the pseudosphere

be contradictions regarding properties of an ordinary Euclidean surface of revolution. With this insight, what Beltrami showed is often paraphrased by saying that hyperbolic geometry is "just as consistent as Euclidean geometry." Technically, this is known as a *relative consistency proof.* Stated more formally, what he proved is that *Euclidean geometry is consistent if and only if hyperbolic geometry is consistent.*

However, there is a deficiency of the pseudosphere model of hyperbolic geometry in that the pseudosphere does not represent *all of the hyperbolic plane*. Hence it's somewhat awkward to say how Postulate II is supposed to hold since geodesics cannot be extended indefinitely as they tend toward the points on the boundary circle at the bottom of the pseudosphere. Beltrami also addressed this issue and developed another model of hyperbolic geometry. This model was also studied by the German mathematician Felix Klein (1849-1925) and is shown in Figure 2. It is more commonly known as the *Klein model* for this reason, but Beltrami really deserves some of the credit too. In this model,

- The points are the points (strictly) inside the unit disk in $\mathbb{R}^2$, and
- the geodesics are chords of the boundary circle.
- The boundary circle is not included, though, so the geodesics are *open* line segments.
- Distances and angles are defined in terms of a "non-standard" Riemannian metric.

As on our example surface $\mathbb{H}$, this means that the lengths of curves in this model differ from the lengths of the corresponding Euclidean curves. *Unlike*

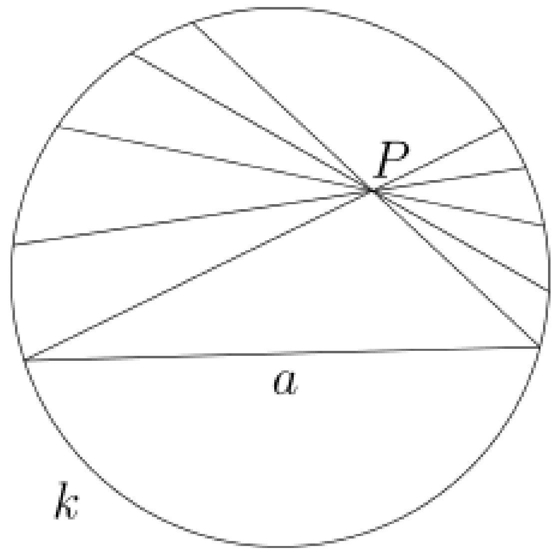

Figure 2: The Beltrami-Klein model of the hyperbolic plane

the case in our example surface $\mathbb{H}$, angle measurements are *also different*. This means, for instance, that "right angles" in this model do not look like Euclidean right angles. The fact that they do in our model $\mathbb{H}$ is a special property known as *conformality*.

The model we studied in our discussion was developed by a different mathematician in what started out as a very different context. One of the great anecdotes about mathematical inspiration comes from the French mathematician Henri Poincaré (1854 - 1912). At one point, Poincaré was deeply involved in trying to solve a problem in complex analysis regarding what he called "Fuchsian functions"–functions defined on the unit disc in the complex plane:

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

that would satisfy particular types of transformation rules.

Poincaré says:

Just at this time I left Caen, where I was then living, to go on a geological excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of non- Euclidean geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake I verified the result at my leisure.

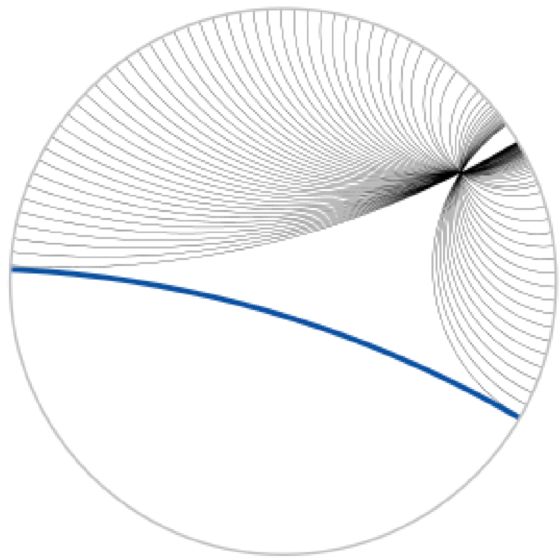

Figure 3: The Poincaré model of the hyperbolic plane

In the *Poincaré disc model* shown in Figure 3[1] the points are again the points in the interior of the unit circle. But now geodesics are arcs of circles meeting the boundary circle at right angles. In the figure above, we see infinitely many such geodesics through a point not lying on the solid blue geodesic. The parallels are the two circles meeting the solid blue curve at points on the boundary circle. (Those points are not points in the hyperbolic plane though; we would think of them as "points at infinity" and the parallels are merely asymptotic in the hyperbolic plane itself.)

As a domain for defining functions of a complex variable, the upper half plane we studied in the final small group discussion days is actually equivalent to this disk, via a mapping that can be described in complex coordinates by

$$\phi : z \mapsto \frac{z-i}{z+i}$$

(This maps $i \leftrightarrow (0,1)$ to 0, the x-axis to the unit circle and the upper half-plane to the interior of the unit circle.) As a result he saw how to create *analytical models* of hyperbolic geometry using the interior of the unit disc and the upper half-plane in the complex plane, and do all of the hyperbolic constructions we did in terms of complex analysis(!)

[1] I believe Beltrami (as well as Riemann) also defined something equivalent to this version, and well before Poincaré did. Why isn't it named after one of them? The best answer is probably that mathematicians are generally very bad historians as you might guess already from the Beltrami-Klein case discussed before(!) Things tend to get named after more famous mathematicians or after the place most people learn about them, not after the first people to develop them. Sometimes also connections are overlooked or not appreciated if constructions are written down in different ways.

So we have reached the end of the part of our journey where I serve as guide. I hope you have found it an interesting trip! Starting from the beginnings of geometry as a deductive science in the *Elements* of Euclid, continuing in the attempts to free Euclid of perceived flaws (especially regarding the status of Postulate V), through Bolyai and Lobachevsky and their flights of imagination, then the development of geometry through the introduction of coordinates and tools from calculus, we have seen how the history of the subject has been shaped by what went before. I hope you have a bit more of an appreciation of how the mathematics of the present connects with the questions that began the study of the subject, and how mathematics is really a very unified subject, where the neat divisions into algebra, analysis, and geometry are actually pretty arbitrary. When it comes to understanding the features of the mathematical world, any tool is potentially useful and it's really "all of a piece."