MATH 392 – Geometry Through History
Geometry of the Hyperbolic Plane, I
February 19, 22

If, following Gauss, Lobachevsky, and Bolyai, we seek to understand the geometry obtained by assuming Postulates I - IV, (and the axioms of continuity, Pasch's Axiom, etc.) and Saccheri's HAA, then the resulting geometric structure is called the *hyperbolic plane*, and we are doing *hyperbolic geometry.*

Summary to this point:

- Given any line $\overleftrightarrow{AB}$, and a point P not on $\overleftrightarrow{AB}$ with $PA \perp \overleftrightarrow{AB}$, there are *infinitely many* lines through P not meeting $\overleftrightarrow{AB}$
- Given B, among the previous lines there is a special one $\overleftrightarrow{PQ}$ called the *parallel in the direction of* B which is distinguished by the fact that any line $\overleftrightarrow{PS}$ with S in $ABQP$ with $\angle APS < \angle APQ$ meets $\overleftrightarrow{AB}$ on the ray containing B. On the other hand, if S lies on the same side of AP as B and $\angle APQ \le \angle APS \le 90°$, then $\overleftrightarrow{PS}$ does not meet $\overleftrightarrow{AB}$. Intuitively, the parallel separates lines through P that do meet $\overleftrightarrow{AB}$ on the same side of A as B from the lines that do not meet $\overleftrightarrow{AB}$.
- The parallelism relation on lines is *symmetric* and *transitive*
- When $\overleftrightarrow{PQ}$ is the parallel in the direction of B, $\angle APQ = \Pi(AP)$ is called the *angle of parallelism*
- $\Pi(AP)$ depends only on the length of AP, not on the locations of the points A and P. (In particular, it is the same on both sides of AP.)
- $\Pi(AP)$ is a *monotone decreasing function of the length* AP
- An immediate consequence of the last is the *counterintuitive-sounding fact* if you are used to thinking in Euclidean terms (relying on Postulate V for intuition): *parallel lines have no common perpendiculars* – if a segment AP is perpendicular to one of a pair of parallel lines, then it makes the angle of parallelism $\Pi(AP)$ with the other one, and that angle is *always acute*(!)

Pencils of lines:

In the geometry of the hyperbolic plane, the following collections of lines called *pencils* will be important. (Note: a "pencil" of geometric objects is a slightly old-fashioned general term for a *one-parameter family* of those objects.)

1. Given a point P, the collection of *all lines containing* P is called $\mathbf{P}_P$

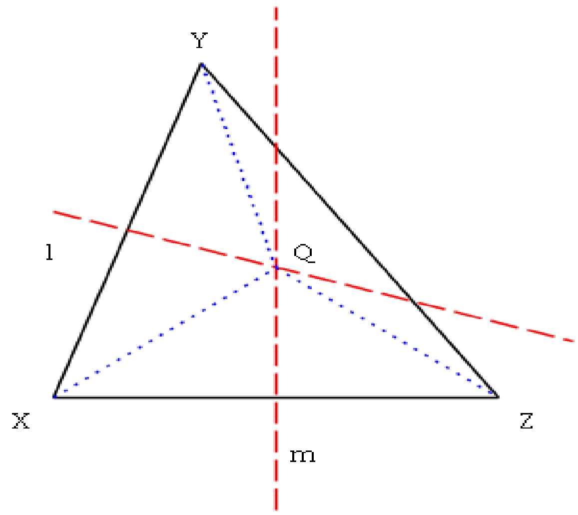

Figure 1: Figure for Theorem 1, Case I.

2. Given a line $\overleftrightarrow{AB}$, the collection of *all lines meeting* $\overleftrightarrow{AB}$ *at a right angle* is called $\mathbf{P}^{\perp}_{\overleftrightarrow{AB}}$.

3. Given a line ℓ, the collection of *all lines parallel to* ℓ is called $\mathbf{P}_\ell$.

Theorem 1 (Theorem 4.4 in McCleary) *Let X, Y, Z be three distinct points, determining the triangle ΔXYZ. Let ℓ, m, n be the perpendicular bisectors of the sides XY, XZ, YZ respectively. Then those three lines are always in a pencil of one of the three types above.*

- Case I: Suppose two of the lines, say ℓ, m actually intersect at some point Q.
- Then $XQ = YQ = ZQ$, so Q is the center of a *circumscribed circle* about the triangle ΔXYZ. (Why?)
- Then Q must also lie on the remaining line n. (Why?)
- Hence in this case, all three lines are in the pencil $\mathbf{P}_Q$.
- Note: In Euclidean geometry, *every triangle* falls into this case. Do you see why?

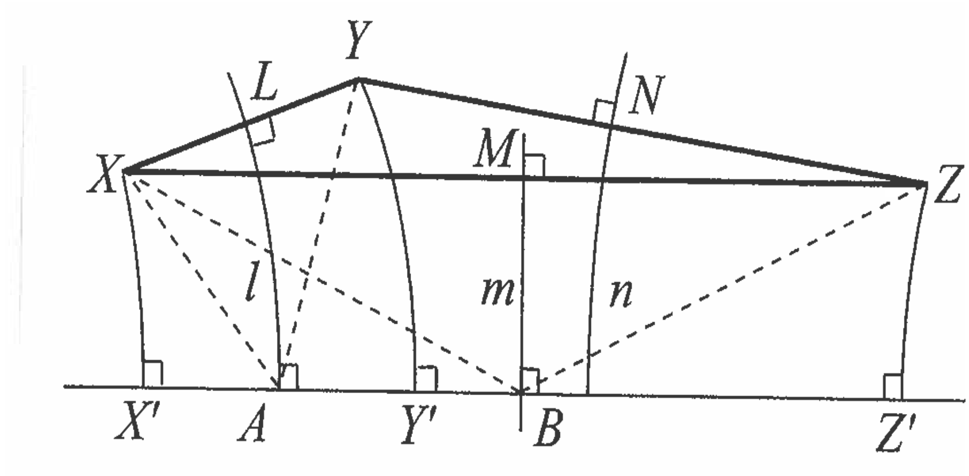

Figure 2: Figure for Theorem 1, Case II.

- Case II: Suppose two of the lines, say ℓ, m have a common perpendicular, the line $\overleftrightarrow{AB}$, where ℓ meets that line at A and m meets that line at B (both at right angles, of course!) (Note that this implies they cannot meet, so we're not in Case I.)
- Let L, M, N be the midpoints of the sides XY, XZ, YZ respectively (see figure)
- Drop perpendiculars XX', YY', ZZ' to $\overleftrightarrow{AB}$.
- Join XB and ZB. Then $\Delta BMZ \cong \Delta BMX$ (Why?) and $\Delta XBX' \cong \Delta ZBZ'$ (Why?)
- Hence $XX' = ZZ'$.
- Similarly, joining XA and YA we get $XX' = YY'$. Hence $YY'ZZ'$ is a Saccheri quadrilateral. The bisector n of the top side YZ meets the bottom side $Y'Z'$ at a right angle at its midpoint by a fact we proved before.
- Hence in this case, all three lines are in the pencil $\mathbf{P}^{\perp}_{\overleftrightarrow{AB}}$.

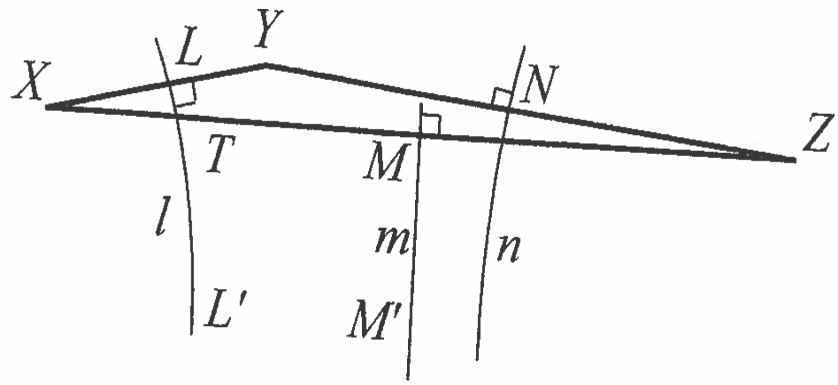

Figure 3: Figure for Theorem 1, Case III.

- Case III: If we are not in Cases I or II, then no pair of the lines $\ell = LL', m = MM', n = NN'$ meet, and no two of them have a common perpendicular. This implies that each pair is parallel. But we have to show that ℓ, m, n are all *parallel in the same direction*, and this comes down to showing that $\angle L'TZ$ and $\angle XSN'$ are acute so they are the corresponding angles of parallelism.
- For $\angle L'TZ$, acuteness follows by this argument: ΔXTL is a right triangle, so the exterior angle $\angle ZTL$ is obtuse ($> 90°$). Then $\angle L'TZ$ is $180° -$ (an obtuse angle), so it must be acute.
- The argument for $\angle XSN'$ is similar.

Additional Definitions

- Given a pencil $\mathbf{P}$ of lines (of any one of the three types), we say two points X and Y *correspond with respect to* $\mathbf{P}$ if the line of the pencil that passes through the midpoint of the line segment $\overline{XY}$ meets that segment at a right angle.
- *When* $\mathbf{P} = \mathbf{P}_Q$ *is the pencil of lines through* Q, then set of all points corresponding to a given point X is what familiar curve? It's the ____.
- *In the case* $\mathbf{P} = \mathbf{P}_\ell$ *is the pencil of lines parallel to some line* ℓ, the set of all points corresponding to a given point X is called a *horocycle* through X. What curve in Euclidean geometry has the defining property of a horocycle? It's the ____. In hyperbolic geometry, horocycles are not lines, but they can be thought of as "circles with center at infinity.")

- *In the case* $\mathbf{P} = \mathbf{P}_m^{\perp}$ *is the pencil of lines perpendicular to some line* m, the set of all points corresponding to a given point X is the *curve of points equidistant from the line*. (This follows from a property of Saccheri quadrilaterals we proved previously. The curve of points equidistant from a line is *not another line* in the hyperbolic case(!)